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THE ANTECEDENTS OF ALGEBRA

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Dedicato a Isabella,
che si è presa cura di me
in un momento in cui
ci avevo bisogno,
con gratitudine

What follows is an outline of the prehistory of algebra that differs fundamentally from those found in general histories of mathematics. It builds on recent reexaminations of a large number of sources which have revealed connections and dissimilarities not noticed so far[1]. Part of the story can be read out rather directly from the sources; part of it, however, only follows from combination of evidence of many kinds.

A presentation of the final outcome of the analysis which were supported point by point by the documentary evidence might easily become obscure, or at best present itself as a methodological exercise[2].Instead I have chosen to present first the synthetic picture which arises, and to introduce and discuss the details of the sources only afterwards.

I. Subscientific mathematics

First of all, however, I shall have to present a concept of general validity for the understanding of the pre-Modern world: "subscientific" knowledge, in particular "subscientific" mathematics.

Subscientific mathematics was a mathematical practitioners', whence a specialists' possession, and thus not to be regarded simply as "popular" or "folk" mathematics – to do so would be to repeat the contempt of the scholars of the epoch for everything not scholarly or aristocratic as

[1] With many interruptions, I have been engaged in this reexamination since 1982, at first working on Old Babylonian material, later also on sources from the Islamic Middle Ages. The reconstruction which is set forth in the following dawned on me in 1989 as a possibility and was presented briefly as a hypothesis in [Høyrup 1991a: note 69] (earlier publications of mine refer to a direct connection between the Old Babylonian school tradition and the medieval mensuration texts, which closer analysis made implausible). Only work on Leonardo Fibonacci and Luca Pacioli in 1992 put me on the trace of evidence that allowed confirmation and elaboration of this hypothesis.

[2] This kind of exposition can be found in [Høyrup 1993].

indistinct *populus* and thereby to blind oneself. But the specialists who carried it were lay, did not belong to any school tradition, nor to any scientific or philosophical environment. They were accountants, surveyors, architects, and the like[3]. And they were taught as apprentices by other practitioners from the field, not by professional teachers of any kind.

It may seem near at hand but is indeed misleading to regard the subscientific category as nothing but another name for "applied mathematics". It is misleading for two reasons. Firstly, the concept "applied mathematics" presupposes that it is the mathematics of theoreticians which is transferred and applied to practical problems. This is how we are accustomed to understand the relation between the different kinds of mathematics in our own world, and if we are willing to take into account that the transfer process may involve considerable reshaping (not so much of results as of structure and justifications) it is grossly a true picture. The practitioners of the pre-Modern world, however, had *their own* mathematics. Results obtained by scientific mathematicians might be adopted, as was the Archimedean approximation to π in certain early medieval traditions (by way perhaps of the Heronian attempt to make *real* applied mathematics); but the pace was extremely slow, and the process uncertain and unsystematic.

Secondly, subscientific mathematics itself contained a "pure" level. This kind of mathematics is well known by historians of mathematics as "recreational mathematics", but its nature and social appurtenance is veiled by this misnomer and by the nature of our sources for this kind of mathematics.

The main function of the "recreational" problems within the practical professions was to allow the display of professional valour. Their format is that of the riddle, and as other riddles in the oral culture (we may think of the riddle of the sphinx, or the function of riddles in so many folk tales) they were meant as challenges. But the challenge regarded professional competence, not general shrewdness. At times implicitly, at times explicitly their question is an injunction: "If you are worth anything *as a calculator*,

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[3] I discuss the concept in depth in [Høyrup 1990a], including however traditions like that of the Old Babylonian scribes as a particular, "scholasticized" subtype. For reasons that are in part reflected in the present paper, I now regard this as an unhappy conceptual conflation.

tell me ...".

Much of what mathematical practitioners would do in their daily activity was pretty trivial; how to find the area of a rectangular field can be learned by any bungler. Ability to solve such problems would not prove the worth of anybody. The riddles, for this reason, have to deal with something more complex than what was known from daily use – that is, with problems of *no use*, problems that are *pure* as far as content is concerned (if meant to be relevant for *professional* esteem, on the other hand, the form could only be that of practical problems, i.e., only be *applied*).

Generally our sources for pre-Renaissance mathematics are either theoretical treatises or schoolbooks (the latter category embraces for instance the Egyptian and Babylonian mathematical texts). Sources for the real uses of mathematics (in particular the use by non-scholars who might be il- or semiliterates) are extremely rare – and *when* they are found, the numbers and drawings they contain rarely allow us to conclude much about the techniques that were used.

The subscientific traditions only become visible in four kinds of sources. (i) Their material was sometimes adopted into literate traditions (in the following we shall meet several examples), and sometimes the subscientific core can be extricated. (ii) Occasionally, scientific mathematicians undertook a critique of what was done incorrectly or incompletely by the practitioners[4]. (iii) Literates might sometimes create problem collections where the "recreational problems" really served as such[5]. (iv) In societies with widespread literacy, finally, handbooks might be written (by practitioners or by writers close to their environment) for other practitioners[6].

Handbooks of the latter type may (as a rule, do) describe techniques of real use. But sources of types i–iii tend naturally to disregard what seems trivial and – for this very reason – to inform us only about the complex,

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[4] For instance by Abū Kāmil, who found the complete set of solutions to the indeterminate problem of the "hundred fowls" [ed. Suter 1910].

[5] Typical examples are the monastic problem collections of the Late Middle Ages – see [Folkerts 1971] – and the arithmetical epigrams of the *Anthologia graeca* XIV. Even the *Propositiones ad acuendos iuvenes* (see below) belongs to the genre.

[6] Typical examples are the *Mišnat ha-middot*, Savasorda's *Liber embadorum*, the *libbri d'abbaco* of the Italian Late Middle Ages, and even a printed book like [Rudolff 1540].

"pure" or "recreational" level of subscientific mathematics. This is the reason that we are much better informed about the "pure" than about the practical aspect of subscientific mathematics; but the use which is made of the problems in these sources also hides their original function and make them appear as mathematical entertainments. Usually their eristic character is only visible in sources of type iv.

Another characteristic is visible in sources of all types. Not only are the problems "applied in form", i.e., apparently concerned with affairs of everyday, and "pure in content", i.e., actually concerned with questions that would never present themselves in real life; they also invariably carry some striking or absurd feature – one hundred monetary units buy exactly one hundred animals, a camel transporting grain will eat exactly all the grain unless a clever trick is applied[7], etc. This has to do with the oral setting of these as well as other riddles: a riddle is more easily remembered and a better challenge if it is striking (once again, the riddle of the Sphinx may serve as a paradigm, with its combination of four, two and three legs with the most significant moments of the day).

All this has little to do with algebra proper. As we shall see, however, the whole prehistory of algebra is strongly involved with subscientific traditions and in particular with recreational problems. One may even assert that *algebra* emerges precisely in the process where proto-algebraic techniques are disentangled from the "recreational" setting.

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[7] Since the example will serve later, I quote the problem in full:

> A paterfamilias had a distance from one house of his to another of 30 leagues, and a camel which was to carry from one of the houses to the other 90 measures of grain in three turns. For each league, the camel would always eat 1 measure. *Tell me, whoever is worth anything*, how many measures were left.

The problem is N° 52 (version B) in the Carolingian collection *Propositiones ad acuendos iuvenes* ([ed. Folkerts 1978: 74] – emphasis added). As everywhere in the following where nothing else is stated, the translation is mine.

The solution told in the text is that the camel makes an intermediate stop after twenty leagues, returns to take another load, repeats the intermediate stop and return, and finally carries the remaining load in one turn. As it is easily seen, the solution is not optimal (two intermediate stops, after 10 and after another 15 leagues, saves extra 5 measures); but such things are not asked for when problems serve as riddles.

II. The story

Subscientific beginnings

Already in the later third millennium B.C., Mesopotamian surveyors appear to have known and used the rule[8]

$$\square(R-r) = \square(R)-2\sqsubset\!\sqsupset(R,r)+\square(r) ,$$

and to have known that the square on the bisecting transversal of a trapezium is the average between the squares of the parallel sides. A very simple geometrical argument can be given for this, based on a kind of area geometry which we shall meet below (I leave to the reader to reconstruct it from Figure 1). But there is no evidence that any other proto-algebraic knowledge was present.

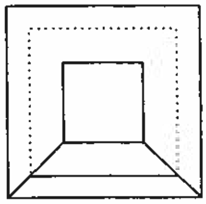

Figure 1.

Shortly after the turn of the millennium, however, Akkadian-speaking surveyors in Middle Iraq knew the trick of the quadratic completion, and used it in a number of recreational riddles more or less as the intermediate stop of the camel[9]. If Q designates the area of a square; s the corresponding side (Q_i and s_i, $i = 1, 2$, when two squares are involved); and $_4s$ "all four" sides of the square, the following problems circulated in the environment: $s+Q = 110$ and $_4s+Q = 140$; probably also problems with differences (area minus side(s), and side(s) minus area) and questions about the diagonal when the side is given, and vice versa. For two squares, $Q_1+Q_2 = \alpha$, $s_1\pm s_2 =$

[8] [Whiting 1984: 65f]. Here and in the following, $\square(s)$ stands for the (geometric) square with side s, and $\sqsubset\!\sqsupset(l,w)$ for the rectangle with length l and width w.

[9] Everything which is said about these surveyors, and about their knowledge and techniques, builds on reconstruction from indirect evidence (discussion of which follows in chapter III). They may not have been scribes, perhaps they were even illiterate (however, not innumerate); they may also have been taught as scribes, but then the problems and techniques in question belonged to the oral lore of their profession. Not only as far as the absence of written sources is concerned but also from a cultural point of view, the environment was functionally non-literate.

β and $Q_1-Q_2 = \alpha$, $s_1 \pm s_2 = \beta$[10]. Concerning rectangles (area A, length l, width w, diagonal d): $A = \alpha$, $l \pm w = \beta$; $A+(l \pm w) = \alpha$, $l \mp w = \beta$; $A = \alpha$, $d = \beta$.

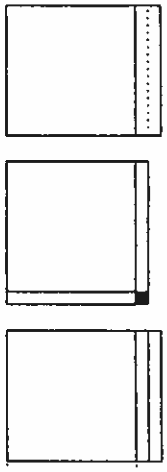

Figure 2.

The problems were solved by geometrical cut-and-paste procedures – the first two steps of Figure 2 show the solution of the problem $s+Q = 110$ (in distorted proportions): The side is understood as provided with a width 1, and thus as a rectangle ⊏⊐(1,s) naturally located along the square. This rectangle is bisected, and its outer half moved around so as to yield a gnonom of area 110. What is lacking if this gnomon is to be completed as a square is the black square □(½). When it is added, the total area becomes 110¼, from which the side of the completed square is found as 10½. When we remove the rectangle ⊏⊐(½,s) that was attached to the lower side of the original square, we see that the side of this square must be 10½–½ = 10. If the problem is translated into the equation $x^2+x = 110$ and solved by quadratic completion, the two solutions run completely parallel; moreover, just as our modern algebraic procedure is "analytical" by treating x as if it were a known number, so is the way the geometrical solution deal with its "unknowns". We may characterize the problem and its solution as "quasi-algebraic"

The same diagram shows the solution of the rectangle problem $A = \alpha$, $l-w = \beta$, with the only difference that in this case the rectangle which is removed below has to be brought back to its original location so as to restore the original rectangle. Even the square problem $Q-s = 90$ is solved like this.

[10] Most likely, such problems referred to a standard rectangle with fixed length and width, just as the problems dealing with a square took its side to be 10. The sources, however, tell us nothing with certainty, even though some suggest $l = 8$, $w = 6$, $d = 10$. Similarly, two-square problems probably operated with standard squares. α and β should thus not be read as numbers which might be varied, only as numbers which cannot be identified.

This may seem strange to us, but fits well with what else we know about "recreational mathematics": The problem of the "hundred fowls" carries this name because it almost invariably deals with 100 monetary units and 100 animals (mostly fowls); and for 2500 years, all problems about repeated doublings had 30 repetitions; then the chess-board variant with its 64 doublings entered the scene (see [Høyrup 1987: 287f]), since when these two version coexisted for another 500 years at least.

The problem $_4s+Q = 140$ was solved in a slightly different manner (see Figure 3: each of the four sides is regarded as a rectangle ⊏⊐$(1,s)$, located once again where they belong "naturally"; the area 140 is hence a cross-shaped configuration. Presumably it was completed by a square □(1) in each corner, yielding a completed square of area 144 (but possibly one fourth of the configuration was regarded alone – our indirect sources disagree on this point).

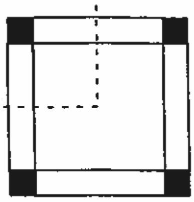

Figure 3.

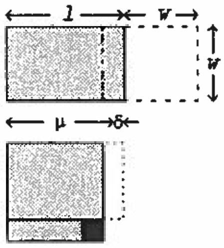

Figure 4.

Problems of the type $A = \alpha$, $l+w = \beta$ (and $s-Q = \alpha$ or $_4s-Q = \beta$, if they were present) were solved according to the scheme shown in Figure 4: The rectangle corresponding to the excess (δ in the following) of the length over the average (μ) of length and width is cut off and moved so as to create a gnomon of known area (that of the rectangle). Since also the area of the completed square is known (*viz* □($^{\beta}/_{2}$)), even the side of the completing square (and thus the width of the rectangle which was moved) is known.

The problems $A+(l\pm w) = \alpha$, $l\mp w = \beta$ were reduced to the types $A = \alpha$, $l\mp w = \beta$ by a geometric "change of variable" – Figure 5 shows how it was done in the case $A+(l-w) = \alpha$, $l+w = \beta$: as usually, the segment $l+w$ is thought of as provided with a width 1; similarly for l and w. The geometrical aggregate of $A+(l-w)$ and $l+w$ is thus a rectangle of known area (*viz* $\alpha+\beta$), while its width is $w+2$ and its length l. Even the sum of its sides is thus known (*viz* $\beta+2$).

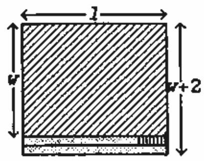

Figure 5.

The final problem type concerning a single rectangle of whose presence we can be fairly certain was $A = \alpha$, $d = \beta$. Even in this case, the problem was reduced to the type $A = \alpha$, $l-w = \beta$, but apparently by means of a synthetic rather than an analytical argument – see Figure 6, where the rectangle is present in four copies: *LF*, *FH*, *HJ*, *JL*, each of which are bisected by the diagonal. If, from the square on the diagonal that results, twice the area (four times the semi-area) is removed, we are left with the square on

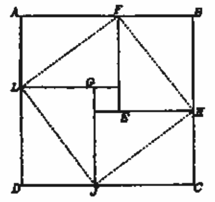

Figure 6.

the difference between the length and the width, from which this difference can itself be found[11]. We may observe that the diagram is related to a familiar "naive" proof of the Pythagorean theorem (also known from Chinese sources), and that the solution of the problem presupposes knowledge of this principle.

Problems $Q_1+Q_2 = \alpha$, $s_1\pm s_2 = \beta$ appear to have been solved by means of a variant of Figure 6 – see Figure 7. It is deduced without difficulty that twice the sum of the areas (e.g., *AN+MC+PB+DQ*) exceeds the square on the sum of the sides by the square on their difference. Actually, the surveyors will rather have operated with the average μ (*AR*, the semi-sum) of the sides and their deviation δ from this average (*ER*, the semi-difference), as suggested by the broken line. We may notice (but probably should not make too much of the observation) that with this addition to the diagram, all the transformations of Figure 2 and Figure 4 are incorporated. In both cases, the unknown rectangle is represented (say) by *AP*, the gnomon into which it is transformed by *ARUMVTA*, the completing square by *MS* and the completed square by *AN*.

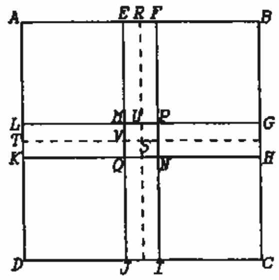

Figure 7.

Sources do not allow us to determine how problems $Q_1-Q_2 = \alpha$, $s_1\pm s_2 = \beta$ were solved. They do tell, however, that the smaller square was thought of as concentrically embedded in the larger one – see Figure 8 – and we may imagine that the band between the two squares was noticed to be dissectible into four gnomons (full lines), each of which equals ⊏⊐(μ,δ), or directly into the four rectangles (broken lines) (but the same rule can also be verified on Figure 7, cf. below).

So far, everything looks as perfect mathematics, though probably "naive", that is, built on what can immediately be "seen" to be true, and not on explicit proofs. But there are unmistakeable traces in the sources

[11] Possibly, the reduction to $A = \alpha$, $l+w = \beta$ was also in use. In this case, twice the area is joined to the square on the diagonal, which gives us the square *ABCD* on the sum of the length and the width.

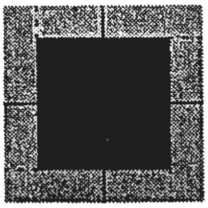

Figure 8.

of one suspicious feature: the square problem $d-s = 14$, with assumed solution $s = 10$, $d = 14$. Since our written sources have eliminated this "error" (as it had to be from their mathematical point of view), we can say no more about the specific topic, only that it tells us not to think of the whole cluster of problems in terms of mathematical *theory*, however much the problems resemble algebra in their use of analytical procedures.

The scribe school

As already told, we have no direct sources for all this. Oldest among the indirect sources are the mathematical texts from the Old Babylonian scribe school. Around 1800 B.C., the Mesopotamian scribe school adapted itself to the situation that Akkadian had become the dominating language and Sumerian had died as a spoken language. The classical literature was now preserved in bilingual versions, new epics were written in Akkadian, and wholly new literary genres arose in Akkadian. One is the omen literature, the other, the one which regards us here, is a new kind of mathematics.

The sexagesimal place value number system (introduced in all probability during the "neo-Sumerian" twenty-first century B.C. as a tool for for intermediate calculations) was taken over from the preceding school tradition, together with the extensive use of tables (for multiplication, of reciprocals, and of technical coefficients). But the quasi-algebraic problems of the surveyors were adopted together with the cut-and-paste technique used to solve them, and second-degree "algebra" occupies about half of the corpus.

The Old Babylonian texts provide us with the evidence for the geometrical technique. This cannot to be seen, it is true, in the traditional translations and commentaries of the text editions (MKT, TMB, MCT, TMS, and scattered publications of single texts), all of which have been based on the assumption that the original geometrical vocabulary has to be read as a set of metaphors for arithmetical operations. It only follows from an in-depth analysis of the structure of the vocabulary, which reveals, for instance, that two operations traditionally identified as "addition" are kept strictly apart – "aggregating *a* and *b*", a symmetric and truly arithmetical operation that allows the addition of all entities provided they possess a measuring number (be it mice and elephants or lines and areas); and "joining *d* to C", a concrete and asymmetric operation where C stays in place and may be said to conserve its identity while absorbing *d* and thus expanding its size. Similarly, "subtractions" are two ("comparing" and "removing"), while "multiplication" splits up into four different operations[12].

But the Old Babylonian texts do not repeat exactly what had been done before the subject became a school discipline. Even the terminological distinction between different additions may have been the outcome of the kind of critical reflection that also expelled the problem $d-s = 4$ from honest mathematical company. The first adoption into the school seems to have taken place in the northern Ešnunna region, which also produced the earliest law code written in Akkadian. The mathematical texts from Ešnunna exhibit several archaic features – e.g., an introductory riddle-like phrase "If somebody has asked you". An important text from Ešnunna [ed. Goetze 1951] also *joins* sides to area, thus showing that it regards the sides as possessing a standard width 1, as currently done by subscientific surveyors until the Renaissance[13]. Later texts take care to *aggregate* in this case, showing thereby that they really regarded the lines as lines and not as rectangles. In order to make possible the solution shown in Figure 2 they therefore provide the side explicitly with a "*waṣītum* 1"[14]. The step looks like a close parallel to Euclid's definition of the line as "a length without width", which may have been meant (when first introduced,

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[12] The over-all analysis of the vocabulary, with full reference to the texts that underlie the argument, is [Høyrup 1990], while [Høyrup 1993b] is a more complete analysis of the subtractive operations and their terminology. The texts do not contain the drawings whose making they refer to; these will have been made in some other medium, probably sand or a dustboard.

[13] This way to look at things astonishes everybody with some mathematical school training, but is in fact what underlies the metrology used in thirteenth-century Pisa when land was bought and sold (*Pratica geometrie* [ed. Boncompagni 1862: 3f]). Luca Pacioli [1523: [II], 6^v–7^r] informs us similarly about fifteenth-century Florence.

[14] From *waṣûm*, "to go out". The *waṣītum* is hence something which "goes/sticks out", protrudes or "projects" – further on to be translated "projection".

fore Euclid's times) to bar the same "mistake"[15].

ns went far beyond conceptual clarification. In agreement
nt of recreational mathematics to the striking, all problems
e confidence be ascribed to the early surveyors deal with
ith an arbitrary multiple of the area; with *the* side or with
ot just "the side taken four times"). The scribe school
ically, replacing the affection for the striking with search
systematic progress (nothing could be more alien to a
). As an example we may list the surviving problems from
xclusively with one or more squares – a tablet of the type
ists the problems but also tells the procedure[16]:

$+s = 45'$
$-s = 14`30$
$-\frac{1}{3}Q+\frac{1}{3}s = 20'$
$-\frac{1}{3}Q+s = 4`46°40'$
$+s+\frac{1}{3}s = 55'$
$+\frac{2}{3}s = 35'$
$Q+7s = 6°15'$
$+Q_2 = 21'40'', s_1+s_2 = 50'$ (reconstructed)
$+Q_2 = 21'40'', s_2 = s_1+10'$
$+Q_2 = 21°15', s_2 = s_1-\frac{1}{7}s_1$
$+Q_2 = 28°15', s_2 = s_1+\frac{1}{7}s_1$
$+Q_2 = 21'40'', \sqsubset\!\sqsupset(s_1,s_2) = 10'$
$+Q_2 = 28'20'', s_2 = \frac{1}{4}s_1$

' was also the basis of fourth-century Athenian metrology is seen in
e three- and five-foot *dynameis* refer to areas equal to three respectively
the usual reading as 3 and 5 *square feet* is anachronistic). Moreover, the
errors that are widespread among Athenians in *Laws* 819e–820a is
f we assume that this thinking was not just a silent presupposition for
l the normal way to think about areas and lines. Without this interpreta-
icult to see any difference between the allegedly different errors – cf.

cribed in the generalized degree-minute-second system, where ', '', etc.
nd `, ``, etc. increasing order of sexagesimal magnitude; 4`46°40' thus
$\cdot 40\cdot 60^{-1}$. In the original, no such marking of the order of magnitude
n was a pure floating-point notation, and on the tablet the number is

13901 (ed. [MKT III, 1–5], translation and analysis in [Høyrup 1992:

14. $Q_1+Q_2 = 25'25''$, $s_2 = {}^2\!/_3 s_1+5'$
15. $Q_1+Q_2+Q_3+Q_4 = 27'5''$, $(s_2,s_3,s_4) = ({}^2\!/_3,{}^1\!/_2,{}^1\!/_3)s_1$
16. $Q-{}^1\!/_3 s = 5'$
17. $Q_1+Q_2+Q_3 = 10`12°45'$, $s_2 = {}^1\!/_7 s_1$, $s_3 = {}^1\!/_7 s_2$
18. $Q_1+Q_2+Q_3 = 23'20''$, $s_2 = s_1+10'$, $s_3 = s_2+10'$
19. $Q_1+Q_2+\square(s_1-s_2) = 23'20''$, $s_1+s_2 = 50'$
20. [missing; $Q_1+Q_2+\square(s_1-s_2) = 23'20''$, $s_1-s_2 = 10'$?]
21. [missing]
22. [missing]
23. ${}_4s+Q = 41'40''$
24. $Q_1+Q_2+Q_3 = 29'10''$, $s_2 = {}^2\!/_3 s_1+5'$, $s_3 = {}^1\!/_2 s_2+2'30''$.

Several observations with bearing on our topic can be made on the basis of this list. Already if one examines the solutions, N° 23 is the sole problem dealing with a single square[17] to have conserved the side 10 – even if 10 has been moved to the order of minutes. (N° 23, moreover, is quite untypical in other respects too, to which we shall return.) Instead, the standard square has the side 30′ (or 30).

More decisive, however, is the whole sequence of problems. We start again with the sum of side and area, mentioning however the area before the side in the statement, while the surveyors had told the side first. This apparently insignificant detail can be interpreted as a reflection of the changed status of the problems[18]: when a riddle is told, one starts with the entity that is immediately known, which for a practising surveyor is of course the side; next comes the derived entity, that is, the area[19]. When

[17] Problems about several squares are submitted to other constraints: ratios and relative differences should be 4, 7, 11, 13, 17, or 19, and ${}^1\!/_4$, ${}^1\!/_7$, ${}^1\!/_{11}$, etc.; absolute differences should be 5 or 10 [Høyrup 1993a]. These constraints – which expand a system whose roots are already visible in Sumerian mathematical texts from the mid-third millennium, rather than continuing the habits of the surveyors – explain that N^{os} 13, 15, and 18 have 10 as the side of their smallest square.

[18] Evidently, this interpretation requires that no other constraints are present, e.g. from grammatical structures; as a consequence it does not apply in subtractive statements.

[19] The equally subscientific *al-jabr* tradition, to which we shall return, mentions the "possession" (*census* in the Medieval translations, normally interpreted as x^2) before the "root" (x in the same interpretation). But here the "possession" is really to be thought of as an unknown amount of money, and the "root" as its square root (y for "possession" and $\sqrt{y}$ for "root" would thus be a more adequate translation); the same psychological "law" thus applies.

In first-degree problems, where we have not been conditioned otherwise by the habits of school mathematics, our own psychology agrees so fully with the law that we do not notice the riddle becomes a mathematical problem, on the other hand, the tendency is to shape the statement in agreement with the solution – and here (see Figure 2), the square area is in place first, and the side can only be joined to it when it has been provided with a "projection".

The next problem (side removed from area) is likely also to be a borrowing from the traditional corpus of riddles. Then, however, comes a sequence of problems with "unnatural" coefficients, several of them even non-normalized. As we remember, the riddles invariably dealt with *the* area, which guaranteed that the problems were normalized; and with *the* or *all four* sides.

Non-normalized problems cannot be solved by cut-and-paste methods alone, and thus called for the introduction of a new technique. As an example we may follow the solution of N° 3 – see Figure 9: From the area $\square(s)$ one third (grey, to the left) is removed, leaving a rectangle $\sqsubset\!\sqsupset(s,\frac{2}{3}s)$; to the right, a "projection 1" is situated, $\frac{1}{3}$ of which together with the side holds a (hatched) rectangle $\sqsubset\!\sqsupset(\frac{1}{3},s)$. In order to obtain a normalized situation (square with attached rectangle), the vertical scale is reduced with the same factor as the width of the square, i.e., with a factor $\frac{2}{3}$. This leaves us with a familiar situation: a square $\square(\sigma)$ with an attached rectangle $\sqsubset\!\sqsupset(\sigma,\frac{1}{3})$, where $\sigma = \frac{2}{3}s$ – and the rest goes as in Figure 2.

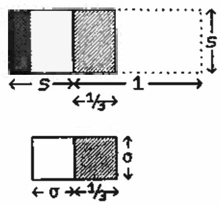

Figure 9.

After a number of variations on this pattern come, as N^os^ 8 and 9, two of the inherited two-square problems (the two which reduce trivially to the first degree are not included here but found in the equally Old Babylonian tablet TMS V); the actual steps of the solutions (a halving of the sum of the areas) suggest that they made use of the rule

its effect: we, no less than Ahmes the copyist of the Rhind Mathematical Papyrus, would find it most awkward if the jug had told that its ninth, the third of its third, its third, and the jug itself three times went into the *hekat*-measure (cf. [Chace et al (eds) 1929: problem 35]); and we, no less than Alcuin the presumed editor of the *Propositiones ad acuendos iuvenes*, would find it stylistically impossible if the man who encounters a group of (36) people had told that the half of their half, together with their half, and twice their number and himself, would have made up 100 persons (cf. [Folkerts (ed.) 1978: 45f]).

$$\square(\mu+\delta) = \square(\mu-\delta)+4\sqsubset\!\sqsupset(\mu,\delta),$$

which is easily verified on the sub-diagram AFNKA of Figure 7 (μ = AL, δ = EL), and from which the average area is seen to be $\square(\mu)+\square(\delta)$.

Then, again, the tablet goes on with variations, now on the two-square theme: in N^os^ 10 and 11 the relative difference between the sides is given, in N° 13 their ratio, and in N° 14 the ratio with excess (to use the idiom of Euclid's *Data*). Of particular interest is N° 12, where the area held by the two sides is given. By means of the diagram in Figure 7, this problem might have been reduced to the case $Q_1+Q_2 = \alpha$, $s_1+s_2 = \beta$, and we might have stayed at the level of naive geometry. Instead, the text calculates $[\sqsubset\!\sqsupset(s_1,s_2)]^2$, which is $\sqsubset\!\sqsupset(Q_1,Q_2)$. The problem is thus reduced to one of the type $l+w = \alpha$, $\sqsubset\!\sqsupset(l,w) = \beta$, where $Q_1 = l$, $Q_2 = w$.

This is one of the great steps in the history of mathematics, one of the very greatest, and whoever feels a chill when faced with intellectual progress should feel it here. What we are confronted with is the earliest documented instance of *representation*. The surveyors' problems, and the preceding problems contained in the present tablet, all manipulated the very entities which were spoken of in the statements. Here, however, a length is taken to *represent* something different from itself, *viz* an area. If any single step demarcates the invention of algebra, this is the one – and since other (slightly later) Old Babylonian texts use lengths and widths to represent pure numbers, prices or complex arithmetical expressions, the step is real, is no mere accident.

Next follow a number of problems dealing with three and even four squares (N^os^ 15, 17, 18, 24), alternating with other types: N° 16 belongs naturally between N° 2 and N° 3 and may have been forgotten at first during copying. N° 19 is, like N° 12, a more genuine extrapolation from N° 8 (we may guess that N° 20 will have been the analogous extension of N° 9).

Too little remains of N^os^ 21 and 22 to allow even guessing at their content. N° 23, however, is an immense surprise (so much so, indeed, that Neugebauer believed it to rest on a scribal mistake that happened to make mathematical sense [MKT III, 14]). We have already noticed the aberrant side of 10´, but this is only one of several anomalies. The statement itself is unusual, first by pointing out explicitly (this is at least the plausible interpretation of the introductory phrase) that it deals with a field, second by mentioning the sides before the area, third by referring to *the four* sides and not to the side taken four times (the pattern of N° 7). The terminology, moreover, deviates from what is found elsewhere in the tablet, imitating apparently the idiom of practical surveying. In contrast to other problems of the tablet (with the possible exception of N° 22), the solution also indicates the unit. Finally, the procedure is not the standard procedure used everywhere else (in this and all other tablets) to solve problems "area plus sides equal number" ("$x^2+\alpha x = \beta$"), i.e., the procedure of Figure 2; instead, the method shown in Figure 3 (with quadripartition) is employed, which is clearly geared to the presence of exactly four sides and awkward in use in other cases (as we shall see, al-Khwārizmī does use it for the general problem, but the outcome only confirms the clumsiness).

The message, in its times, will have been unmistakeable: this is not an ordinary mathematical problem – it is a riddle, and indeed one of the traditional surveyors' riddles. It has, if we regard its location within the tablet, the character of a "last lesson before Christmas". What had once functioned as a challenge had become a piece of truly recreational mathematics; whatever eristic function remained in Old Babylonian mathematics (and there was much, cf. for example [Høyrup 1994]) was now bound up with mathematically more advanced topics. At the same time, the problem had been somewhat adapted to current tastes (as it has happened regularly when other elements of oral culture – folktales etc. – were adopted by literate environments): the side remains 10, but its order of magnitude is adjusted to scribe school habits. The solution by means of quadripartition is also likely to be an innovation (but sources give no certainty), made perhaps because it is more elegant to make only one quadratic completion when it has to be told in terms of the "projection".

The features which characterize the present tablet as a piece of school mathematics – stringency, systematic progress, etc. – recur in other parts of the Old Babylonian corpus. Another tablet of early date containing one of the traditional problems is AO 8862 (ed. [MKT I, 108–111], translation and discussion [Høyrup 1990: 309–320]). The first problem is of the type

$$\sqsubset\!\sqsupset(l,w)+(l-w) = \alpha, \quad l+w = \beta \quad (\alpha = 3`3, \beta = 27),$$

and is dealt with as in Figure 5. Its references to surveying practice are obvious: after telling that he has laid out a field, the speaker states that "I went around it", on which occasion the excess of length over width is joined (without any reference to a "projection" – the standard width 1 is presupposed), after which he "returns". No doubt that we are quite close to the surveyors' "recreational problems". The second problem can be translated

$$\sqsubset\!\sqsupset(l,w)+\tfrac{1}{2}l+\tfrac{1}{3}w) = 15, \quad l+w = 7,$$

and is reduced by the subtraction of ½ $(l+w)$ into $\sqsubset\!\sqsupset(l,w)-\tfrac{1}{6}w$ = 11°30′, and thus into $\sqsubset\!\sqsupset(l-\tfrac{1}{6},w)$ = 11°30′, $(l-\tfrac{1}{6})+w$ = 6°50′. This reduction is close in spirit to the one of N° 1, but the problem itself is clearly of a more sophisticated type than the original surveyors' problems, even though the speaker still "goes around" and "returns". Even the third problem,

$\sqsubset\!\sqsupset(l,w)+\sqsubset\!\sqsupset(l-w,l+w)$ = 1``13`20, $l+w$ = 1`40,

might have been solved in analogous fashion; yet even though "going around" and "returning" still refer us to the wonderland of recreational surveying, the text takes care to demonstrate a different method. Once again, scholastization results, firstly, in extension of the range of problem types dealt with (and again, the introduction of fractional coefficients is part of the game); secondly, it leads to the application of new methods (in the present case, a variant of Figure 7 appears to be used).

Particularly illuminating are the so-called "series texts", canonized sequences of statements which occupy a whole series of tablets (whence their name). One tablet belonging as number 4 to such a series is YBC 4714[20], which contains the following problems:

1. Q_1+Q_2 = 21`40, s_1+s_2 = 50
2. $Q_1+Q_2+Q_3+Q_4$ = 1``30`, $s_1+s_2+s_3+s_4$ = 2`20
3. $Q_1+Q_2+Q_3+Q_4+Q_5+Q_6$ = 1``52`55, $s_1+s_2+s_3+s_4+s_5+s_6$ = 3`15
4. $Q_1+Q_2+Q_3$ = 30`50, $s_2 = \tfrac{1}{7}s_1+15$, $s_3 = \tfrac{1}{2}s_2+5$
5. $Q_1+Q_2+Q_3$ = 1``8`5 (or $Q_1+Q_2+Q_3+s_1+s_2+s_3$ = 1``9`46), {L}
6. $Q_1+Q_2+Q_3+s_1+s_2+s_3$ = 27`50, {L}
7. $Q_1+Q_2+Q_3$ = 1``17`30, {L}
8. $Q_1+Q_2+Q_3+Q_4$ = 2``23`20, {L}

[20] Ed. [MKT I, 487–492], translation and analysis in [Høyrup 1992: 105–140]. The other tablets of the series are lost.

9. $Q_1+Q_2+Q_3+Q_4+s_1+s_2+s_3+s_4$ = [?], {L}
10. $Q_1+Q_2+Q_3+Q_4$ = 1``15`50, {L}
11. $Q_1+Q_2+Q_3+Q_4$ $^{?}[+s_1+s_2+s_3+s_4]^{?}$ = [?], {L}
12. $Q_1+Q_2+Q_3+Q_4$ = 1``36`15, {L}
13. $Q_1+Q_2+Q_3$ = 2``47`5, $s_1$ = 1`20, {L}
14. $Q_1+Q_2+Q_3$ = 2``47`5, s_2 = 45, {L}
15. $Q_1+Q_2+Q_3$ = 2``47`5, s_3 = 40, {L}
16. $Q_1+Q_2+Q_3$ = 2``47`5, {L}
17. $Q_1+Q_2+Q_3$ = 2``47`5, {L}
18. $Q_1+Q_2+Q_3$ = 2``47`5, {L}
19. $Q_1+Q_2+Q_3$ = 2``47`5, {L}
20. [too damaged for reconstruction]
21. $Q_1+Q_2+Q_3+Q_4$ = 52`30, $s_{i+1} = s_i+\frac{1}{7}s_1$
22. $Q_1+Q_2+Q_3+Q_4+s_1+s_2+s_3+s_4$ = 54`20, $s_{i+1} = s_i+\frac{1}{7}s_1$
23. $Q_1+Q_2+Q_3+Q_4$ = 52`30, $s_{i+1} = s_i+\frac{1}{4}s_4$
24. $Q_1+Q_2+Q_3+Q_4+s_1+s_2+s_3+s_4$ = 54`20, $s_{i+1} = s_i+\frac{1}{4}s_4$
25. $Q_1+Q_2+Q_3+Q_4$ = 52`30, $s_{i+1} = s_i+\frac{1}{5}s_3$
26. $Q_1+Q_2+Q_3+Q_4+s_1+s_2+s_3+s_4$ = 54`20, $s_{i+1} = s_i+\frac{1}{5}s_3$
27. $Q_1+Q_2+Q_3+Q_4$ = 52`30, $s_{i+1} = s_i+\frac{1}{2}\cdot\frac{1}{3}s_2$
28. $Q_1+Q_2+Q_3+Q_4+s_1+s_2+s_3+s_4$ = 54`20, $s_{i+1} = s_i+\frac{1}{2}\cdot\frac{1}{3}s_2$
29. Q_1+Q_2 = 48`45, ⊏⊐$(s_1,s_2)$ = 22`30
30. Q_1-Q_2 = ⊏⊐(25 nindan,s_2), {L}
31. Q_1-Q_2 = ⊏⊐(25 nindan,s_2), {L}
32. Q_1-Q_2 = ⊏⊐(25 nindan,s_2), {L}
33. Q_1-Q_2 = ⊏⊐(25 nindan,s_2), {L}
34. Q_1-Q_2 = ⊏⊐(25 nindan,s_2), {L}
35. Q_1-Q_2 = ⊏⊐(25 nindan,s_2), {L}
36. Q_1-Q_2 = ⊏⊐(25 nindan,s_2), {L}
37. Q_1-Q_2 = ⊏⊐(25 nindan,s_2), {L}
38. Q_1-Q_2 = ⊏⊐(25 nindan,s_2), {L}
39. Q_1-Q_2 = ⊏⊐(25 nindan,s_2), {L}

Here, {L} stands for a changing set of linear equations involving the sides of the squares; there are always precisely as many equations as needed, and often they are tediously (and trivially) complex.

Visible in the list of problems is an attempt at pluridimensional organization. The sequence N^{os} 21–28, for instance, is constructed as a Cartesian product around the twin variation of the difference between sides and the alternation between the sum of the areas alone and the sum of areas and sides. This organization in Cartesian products goes back to the very earliest phase of the Mesopotamian school and Mesopotamian literacy, where it characterizes the so-called "profession list", and other series texts contain it in fuller form; a striking example is the sequence C 38–C 53 of YBC 4668[21], which is constructed as a four-dimensional Cartesian product: in an equation that can be translated as $p+\frac{1}{19}(p-q) = A$, the numerator 1 alternates with 2, the denominator 19 with 7, + with –, and the first p with q.

This sequence is also interesting in another respect. p is not the length of a rectangle but the length multiplied by the ratio between length and width, $p = \frac{l}{w} \cdot l$, while $q = \frac{w}{l} \cdot w$. Since $p \cdot q = l \cdot w$, the given area, p and q are easily found by means of the usual cut-and-paste technique combined with an adequate change of scale; but in order to find l and w from this a number of arithmetical operations are needed ($\frac{l}{w}$ is the cube root of $\frac{p}{q}$ and l the square root of $[\frac{l}{w}] \cdot [l \cdot w]$). A text like this thus shows how far beyond the direct connection to (artificial) mensurational computation the "algebra" of the school tradition had moved within the two centuries that separate the earliest Ešnunna texts from the abrupt end of the Old Babylonian period in 1600 B.C., when a Hittite army sacked Babylon and opened the way to a Kassite conquest of the country.

The scribe school did not survive this change. From now on, scribes were taught as apprentices within a "scribal family". Advanced mathematics (among which everything "algebraic") disappears from the archaeological horizon for more than a millennium; when at least a few second-degree problems turn up again in the Late Babylonian period[22], they are clearly connected to a surveyors environment once again (areas, for instance, are now given in seed measures, which makes it impossible to regard them as arithmetical products). Other evidence too shows that the old surveyors' tradition, however much adopted by the Old Babylonian scribe school, had not been swallowed, and that it survived the collapse of the school and the disappearance of school "algebra".

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[21] Ed. [MKT I, 431–432].

[22] In tablets from around 400 B.C. (dating by J. Oelsner, private communication).

Greece

We do not know how widespread was this descendant of the surveyors' tradition, but it will have been known at least in Mesopotamia and the Syrian region, where Greek calculators will have met it. We do not know exactly when the encounter took place first, but oblique references in Plato's works show us that it happened no later than the fifth century B.C. The use of the term *dýnamis* by calculators as well as certain geometers also strongly suggests that geometers of the generation of Theodoros and Hippocrates of Chios knew of the tradition and were inspired by it[23]. *What* they knew, however, is not at all clear, not least because nothing survives from their hands beyond a possibly unaltered Hippocratean fragment on lunes.

Somebody, however, must have known rather much before 300 B.C. This is evident from Euclid's *Elements* II, propositions 1–10, which, in symbolic translation, tell the following:

1. $\sqsubset\!\sqsupset(a,p+q+...+t) = \sqsubset\!\sqsupset(a,p) + \sqsubset\!\sqsupset(a,q) +...+ \sqsubset\!\sqsupset(a,t)$.
2. $\square(a) = \sqsubset\!\sqsupset(a,p) + \sqsubset\!\sqsupset(a,a-p)$.
3. $\sqsubset\!\sqsupset(a,a+p) = \square(a) + \sqsubset\!\sqsupset(a,p)$.
4. $\square(a+b) = \square(a) + \square(b) + 2\sqsubset\!\sqsupset(a,b)$.
5. $\sqsubset\!\sqsupset(a,b) + \square(\frac{a-b}{2}) = \square(\frac{a+b}{2})$.
6. $\sqsubset\!\sqsupset(a,a+p) + \square(\frac{p}{2}) = \square(a+\frac{p}{2})$.
7. $\square(a+p) + \square(a) = 2\sqsubset\!\sqsupset(a+p,a) + \square(p)$; or, alternatively,
   $\square(a) + \square(b) = 2\sqsubset\!\sqsupset(a,b) + \square(a-b)$.
8. $4\sqsubset\!\sqsupset(a,p) + \square(a-p) = \square(a+p)$.
9. $\square(a) + \square(b) = 2[\square(\frac{a+b}{2}) + \square(\frac{b-a}{2})]$.
10. $\square(a) + \square(a+p) = 2[\square(\frac{p}{2}) + \square(a+\frac{p}{2})]$.

Proposition 6 coincides with proposition 5 if only $b = a+p$, we see. Proposition 5 corresponds, however, to the situation where the sum of two segments is known (as in proposition 9, a and b result from the splitting of a line into unequal segments), and where they are therefore drawn in continuation of each other in the proof; proposition 6, to the contrary, is adjusted to a situation where one segment exceeds the other by a given segment p, and the proof thus draws them in superposition. The same

[23] The problem of the calculators' and the geometers' *dýnamis*, of its probable Babylonian affinities, and of its fate in Greek later mathematics, is discussed in [Høyrup 1990b].

relation holds between propositions 9 and 10, while propositions 4 and 7 are similarly but not identically correlated.

It is an old suggestion that this "geometrical algebra" is inspired by the Babylonian discipline, and is indeed a translation of the results of this supposedly purely numerical technique into geometrical language (a translation necessitated by the discovery of incommensurability).

Closer analysis shows, however, that the sequence of propositions has an odd relation to the total corpus of Babylonian "algebraic" texts. Everything is indeed connected to the small group of original riddles, and has the character of quasi-Kantian "critiques" of the naive methods we already know: proofs that what is currently done is indeed correct. Apart from N^os^ 9 and 10, whose proofs are clearly Greek (presupposing among other things the Greek concept of the quantified angle), all proofs fall into two sections. A first part constructs the diagram and proves that its parts are really squares, rectangles, etc., and that what is supposed to be equal is really so. When this is done, the second part performs the usual cut-and-paste operations – no longer "naive", however, because of the first part of the proof.

If we look at the single propositions, proposition 1 turns out to be a justification of the geometrical addition of rectangles which have one side in common, whereas propositions 2 and 3 concern the special cases where sides are subtracted from or added to square areas. Proposition 7 is the rule which on p. 5 was traced to the later third millennium, and proposition 4 is its natural additive companion piece; later evidence, furthermore, shows them to be related to the two parallel solutions of the problem $A = \alpha$, $d = \beta$ (cf. p. 7). Proposition 6 explains the solution of all problems $Q \pm \alpha s = \beta$ (including "the four sides and the area") and $A = \alpha$, $l-w = \beta$, while proposition 5 has a similar relation to rectangular problems $A = \alpha$, $l+w = \beta$ and to $\alpha s - Q = \beta$. In both cases, Euclid's diagrams coincide with those that can be reconstructed from the Old Babylonian procedure descriptions, and which were shown in Figures 2 and 4. Proposition 8, which serves nowhere else in the *Elements*, was probably used to solve problems $Q_1 - Q_2 = \alpha$, $s_1 \pm s_2 = \beta$ (cf. p. 8); as suggested by the Old Babylonian solution, it may also have functioned as a kind of lemma in the solution of the problems $Q_1 + Q_2 = \alpha$, $s_1 \pm s_2 = \beta$ (cf. p. 14; Euclid's diagram coincides with the section

AFNKA of Figure 7), which in full are justified by propositions 9 and 10 (even they are not used further on).

Euclid is not our only witness. In Diophantos' *Arithmetica* I we find a few problems of the second degree (evidently in numerical, not in geometrical formulation) – and all of them belong to what was identified as the original stock of riddles: $A = \alpha$, $l_1 \pm l_2 = \beta$ (propositions 27 and 30); $Q_1 \pm Q_2 = \alpha$, $s_1 + s_2 = \beta$ (propositions 28 and 29).

Diophantos's use of the subscientific inspiration is not too different in style from what the Old Babylonian school had done. He takes over some problems (as he does from other subscientific traditions – a whole sequence of problems from his Book I are indeed stripped versions of cherished "recreational problems"), but expands in depth and width (much more radically but perhaps less systematically than the scribe school). The Euclidean relation to the material, however, is rather different; it does not even solve problems, instead it proves that the rules and procedures which underlie the traditional solutions are authentic. This corresponds to the generally "critical" style of early Greek philosophy ("critical" again in a quasi-Kantian sense) – the attitude that allows us to distinguish the "scientific" literate approach from the "scholasticized" style of the Old Babylonian scribe school. It is not totally alien to the Babylonian style – one example is the probable elimination of the less than authentic idea that the diagonal of the standard square be 14, another the introduction of the "projection" and the concomitant distinction between real lines and rectangles with width 1. But it is not what dominates; the difference, though not absolute, is genuine.

Al-Khwārizmī

The influence of the anonymous tradition was not restricted to inspiration of Old Babylonian "algebra" and Greek "geometrical algebra". It was also important in the shaping of "real" (i.e., our) algebra.

It did not provide the stem, it is true. The earliest treatise about the subject (*al-jabr wa'l-muqābalah*) which we possess was written by al-

Kwārizmī in the early ninth century[24], but the preface leaves no doubt that he writes about a preexisting technique.

This technique, we know from a slightly later treatise written by Thābit ibn Qurrah [ed. Luckey 1941], was the possession of the group of "*al-jabr* people" or "followers of *al-jabr*". They will have been reckoners of some kind, and *al-jabr* itself a subscientific tradition. Combining the evidence offered by al-Khwārizmī and by Thābit, one may deduct that the technique was purely numerical and "rhetorical". It possessed two levels: one of practical use, representing the unknown in first-degree problems by a "thing" (*šay*ᵓ; *res* in the Latin translations, *cosa* in the Italian abacus treatises); and another, useless but "brilliant" according to al-Khwārizmī (i.e., the "recreational" level used to demonstrate professional valour), dealing with second-degree problems. The former, characterized as "regula recta" by Leonardo Fibonacci (*Liber abaci* [ed. Boncompagni 1857: 191, 203 and *passim*]) and identical with the first-degree *arithmós*-algebra of Greek calculators, may not even have belonged to *al-jabr* proper; the latter, the core of *al-jabr* and probably the genuine sense of the term, would reduce complex second-degree problems to simple standard cases ("possession and roots made equal to number", etc., cf. note 19) by means of rhetorical techniques, and solve these by means of standard algorithms deprived of argument: "halve the number of roots, multiply it by itself, add the number, take the square root, subtract half the number of roots; this is the root, and its square is the possession" in the case »possession and roots made equal to number (for which al-Khwārizmī and his followers for some 700 years give the example "a possession and 10 of its roots equal 39". We do not know where this tradition originated. In principle it may of course descend from the surveyors' riddles or from Old Babylonian "algebra", but in this case it has undergone a radical transformation, and no positive evidence

[24] Another treatise on the topic written by ibn Turk may possibly (but need not) be slightly earlier, but only a fragment has survived – see [Sayılı 1962].

Two Arabic editions of al-Khwārizmī's treatise exist ([Rosen 1831]; [Mušarrafah & Aḥmad 1939]). Both of are made from the same manuscript, which (through comparison with Gerard of Cremona's twelfth-century Latin translation of the first part) turns out to have been revised on at least three occasions (see [Høyrup 1991]). Even the modern translations (English [Rosen 1831], Russian [Rozenfeld 1983]) are made from the same manuscript. As far as it goes, Gerard's Latin translation is thus the best source for the original wording.

supports the assumption. Terminological features suggest a connection to India, but the mathematical technique excludes close bonds with Indian high-level mathematics as represented for instance by Āryabhaṭa and Brahmagupta. It is even uncertain whether (and, given Leonardo's separate reference to the *regula recta* long before he comes to *al-jabr*, not too plausible that) the rhetorical techniques of the "thing" and the techniques of "possessions" and "roots" have common origins.

Al-Khwārizmī's undertaking was not merely to write a brief compendium on the "most useful and on the brilliant" aspects of the technique – this was what his employer, the Caliph al-Maᶜmūn had asked for according to the preface – but also to transform it into a "scientific" subject. In this context, rules deprived of proofs were unacceptable. Proofs, however, could be borrowed from another subscientific tradition: the "recreational" tradition of the surveyors, which – as we shall see – was still alive. For the case "possession and 10 roots equal 39", al-Khwārizmī presents us with two different proofs. The first is based on the configuration of Figure 3 (without quadripartition): the possession is represented by a square $\square(s)$, and the 10 roots are distributed as four rectangles $\sqsubset\!\sqsupset(2\frac{1}{2},s)$ along its edges. This gives a solution $s = \sqrt{[39+4\cdot 2\frac{1}{2}^2]}-2\cdot 2\frac{1}{2}$, which is proved to be equivalent with the solution provided by the standard algorithm ($s = \sqrt{[39+5^2]}-5$); only afterwards (and according to certain traces in the text only in a revised version of the treatise, cf. [Høyrup 1991: 15]) another proof based on the configuration of Figure 2 is presented, which gives us the solution of the standard algorithm directly.

The other two cases are "Possession and numbers made equal to roots" and "Roots and number made equal to possession". Even they are provided with geometrical demonstrations of a similar kind. In all these demonstrations, al-Khwārizmī uses letters to identify points and surfaces, obviously inspired by the Greek model though not always in full agreement with the Greek usage. In some of them, moreover, there is a tainting of "critique". All in all, however, the style is clearly "naive", and in particular the characteristic structure of the first demonstration (also more "naive" than the others in its formulations) leaves no doubt that al-Khwārizmī borrowed his technique from the subscientific tradition.

Probably for this very reason, Thābit wrote his treatise on the *Rectifica-*

tion of the Cases of al-Jabr, in which geometrical proofs with direct reference to *Elements* II.6 for the first and the third case and *Elements* II.5 for the second case – without even mentioning the existence of al-Khwārizmī's demonstrations, of whose existence he can hardly have been unaware[25], but which he may not have regarded as real proofs. Abū Kāmil, in his work on the subject (c. 900 C.E.), merges the two approaches, not only presenting geometrical proofs of the algorithms but also geometrical formulations of the problems that al-Khwārizmī had stated arithmetically, reduced to fundamental cases, and finally solved by the standard algorithms. From then on nobody appears to have remembered the separate origins of *al-jabr* itself and the idea of geometric proofs.

Abū Kāmil still sees the "possession" as a basic unknown – he even shows how to find the "possession" directly, without recourse to the "root", and solves problems dealing with the possession and the square on the possession[26]. Gradually, however, interaction with the geometric demonstrations made the term appear as a frozen metaphor for the second power of the basic unknown; in the same process, what had once been "root" in the sense of "square root of the possession" came to be seen as "the root of the equation", i.e., the solution. A further reason for this identification of the "root" with the fundamental unknown will have been the many problems (present already in al-Khwārizmī's treatise) where the "thing" occurs while a problem is reduced to one of the standard cases, after which its second power is identified with the "possession" and the "thing" itself with the "root".

This development was gradual, and may only have reached completion when the Arabic discipline was translated into Latin and the terminology thus came to be seen as purely technical. But then the sources leave no doubt that it had really happened. If we regard Leonardo Fibonacci's *Pratica*

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[25] Thābit was connected, as had been al-Khwārizmī a generation before, to the ʿAbbasid Court, and (at least through his protectors, the banū Mūsā), to the very same "library with academy", the "House of Wisdom".

[26] In the latter case, Levey [1966: 90] "repairs" the text in order to make it agree with the interpretation of the "possession" as a square, replacing "possession" by "root", reading "possession multiplied by itself" as "the square of the square" and replacing it with "the square of the root"; but see the Latin translation [ed. Sesiano 1993: 329f, 331, 337, 363].

geometrie, the *census*, the Latin translation of "possession", is no longer explained to be represented by a square, as in al-Khwārizmī's work; it *is* a geometrical square. In a passage [ed. Boncompagni 1862: 56] which is obviously based on the section of Gerard of Cremona's translation of al-Khwārizmī's work where *number* is told to fall into three classes, *roots*, *possessions*, and simple numbers without any reference to either [ed. Hughes 1986: 233], Leonardo tells the three natures of numbers and their fractions to be *roots of squares*; *squares*; and simple numbers. What had originally been nothing but a complement, used to transform a subscientific technique into *mathematics*, had conquered the discipline from within. For the third time, the anonymous surveyors' lore had imprinted a literate mathematical tradition decisively.

III. The evidence

So far this story was largely presented as a scenario, a postulate. But the sources are there. However, in order to make them speak about a tradition which almost by definition has left no sources itself, they have to be used in combination. The purpose of this combination is double. First, it must show what is shared between the Old Babylonian, the Greek and the Medieval Islamic tradition. Secondly, it should reveal features that are *not* shared but should have been if direct transfer from one literate tradition to the other had taken place.

Liber mensurationum

The pivotal source is a work which was not even mentioned so far: a *Liber mensurationum*, written by one Abū Bakr, probably in the early ninth century C.E. (or, alternatively, closely dependent even in its terminology on writings from this epoch – cf. [Høyrup 1986: 462, 474]). The Arabic original appears so far to be lost, but a Latin translation [ed. Busard 1968] is known which was made in the twelfth century by Gerard of Cremona, an extremely conscientious translator[27].

The treatise deals, one for one, with a variety of geometrical figures: squares, rectangles, rhombs, isosceles trapezia, asymmetric trapezia with two acute angles at the base, right trapezia, asymmetric trapezia with one obtuse angle at the base, triangles of various kinds, circles and circular sections, and various volumes. Each figure is represented by a standard example, squares thus by □(10) (not used exclusively, however), rectangles by ⊏⊐(6,8), rhombs by one possessing side 10 and diagonals 12 and 16, etc.

Beginning with the chapters on trapezia, the main topic corresponds to what we would expect from the title, *viz* real geometrical computation. The earlier chapters contain only little of this – the chapter on squares thus only the computation of (1) the area and (2) the diagonal from the side. It goes on with a sequence of mostly quasi-algebraic problems with a very familiar look:

3. $s+Q = 110$: s?
4. ${}_4s+Q = 140$: s_u?
5. $Q-s = 90$: s?
6. $Q-{}_4s = 60$: s_u?
7. ${}_4s = \frac{2}{5}\cdot Q$: s_u?
8. ${}_4s = Q$: s_u?
9. ${}_4s-Q = 3$: s_u? (Both solutions are given)
10. $d = \sqrt{200}$; s?
11. $d = \sqrt{200}$; Q?
12. ${}_4s+Q = 60$: s_u?
13. $Q-3s = 18$: s?
14. ${}_4s = \frac{3}{8}\cdot Q$: s_u?[28]
15. $Q/d = 7\frac{1}{2}$: s_u?
16. $d-s = 4$: s?
17. $d-s = 5$ (no question, refers to the previous case).
18. $d = s_u+4$: s? (no reference is made to N° 16).
19. $Q/d = 7\frac{1}{14}$: s?, d

Already in this list we notice the repeated presence of "the four sides" of

[27] I have analyzed the treatise in question in various publications; see, e.g., [Høyrup 1993].

[28] The text is either corrupt or intentionally enigmatic (as is indeed N° 50 – the eristic style of subscientific thought has no been completely left behind).

the square, as well as the precedence of the sides over the area[29]. N^{os} 16 and 18 point to the idea that the diagonal of the square is 14, while N° 19 presupposes the approximation 14$\frac{1}{7}$ – no doubt a later stage[30] (in both cases, however, Abū Bakr gives a mathematically correct solution).

Solutions of two types occur; one type, not identified by name but obviously the solution traditionally – whence "naturally" – belonging with these problems, is given for every problem; the other, indicated for some of the problems as a possible alternative, is "according to *al-jabr*". The first turns out to follow the same pattern as the Old Babylonian texts; the second comes very close to what we know from al-Khwārizmī, but details of the terminology shows that Abū Bakr makes use of a more archaic treatise (evidently not by necessity an older treatise, but in any case not significantly younger – cf. also [Busard 1968: 71]).

The sections on the basic method remind of the Old Babylonian texts not only through their mathematical method but also in "rhetorical structure". The problem is stated in the first person singular, past tense, with the single exception that excesses of (e.g.) a length over a width are told in the present tense, third person singular, i.e. as timeless facts, not as something "I" have brought about. After a reference to the method comes a prescription, shifting between the imperative and the second person singular, present tense. At times the statement is quoted as the reason for a particular step, in which case the quotation is preceded by "because he has said"; and at times an intermediate results has to be remembered, in which case the number is followed by the relative clause "which you should commit to memory" (the Babylonian version is "which your head should retain".

But there is one partial divergence from this agreement. With a few significant exceptions, the Old Babylonian problems never refer explicitly

[29] The recurrence of precisely "the four sides and the area" of a square, with this characteristic order of the members and the distinctive value 10 of the side (all to be found even in Luca Pacioli's *Summa de arithmetica* from 1494) is indeed the strongest single argument for continuity.

[30] So does in fact also N° 15: In N° 19, as in general in Arabic, $\frac{1}{14}$ is spoken of as $\frac{1}{2}$ of $\frac{1}{7}$. Indubitably, N° 15 has arisen from a copying error. Similarly, N° 16 is a slightly distorted version of N° 18 (the "subtraction by comparison" of N° 18 is the traditional formulation and the "removal" of N° 16 and its sequel N° 17 a reflection of waning understanding of the original mode of thought).

to the "he" who is quoted: they start directly by the statement, with the implication that it is the master who poses the problem. Only some early Ešnunna texts begin "If somebody has asked you thus:" – and exactly this is the invariable beginning of the problems of the *Liber mensurationum*.

If we compare the total list of quasi-algebraic problems in Abū Bakr's treatise with the corpus of Old Babylonian texts we observe other significant similarities and discrepancies. Firstly, all "coefficients" in the *Liber mensurationum* are "natural": *the area* , *the side* or *the sides*, etc. Secondly, there is no single instance of representation (not even, say, of an area represented by a line, as in BM 13901 N° 12) – except, of course, in the *al-jabr*-solutions, where it is made very clear that the "possession" *represents* the area, etc. The basic method manipulates precisely those entities which define the problem. Finally, the bizarre structure known from AO 8862 N° 1 (cf. p. 15 and Figure 5) recurs repeatedly, in versions with "the two" and with "the four" sides (the length and the width, or two lengths and two widths, respectively).

This partial overlap, together with the suggestive discrepancies, fits the scenario presented in chapter II – but it does not exclude alternative scenarios, in particular not the possibility that the *Liber mensurationum* represent an impoverished descendant of Old Babylonian school "algebra". In order to eliminate this hypothesis we have to take into account the character of the problems that constitute the overlap, and the dating of the Old Babylonian texts where they occur. All their distinctive characteristics show that they belong, as riddles carried by a specialists' profession, within a subscientific tradition. Even this tradition could, in principle, have arisen by dilution of the school tradition. If this was what happened, however, the school problems which show particular subscientific affinities – the introductory phrase of the Ešnunna texts, the characteristic problem types – would be late; instead, they are invariably to be found in the few texts to which stratigraphy or palaeography ascribe with certainty an early date. The subscientific material is connected with the beginning, not with the dissolution of the school tradition (which, instead, is characterized by rather messy anthology texts).

That the subscientic tradition which inspired the scribe school was Akkadian (at least the branch the school masters knew of) follows from a number of concurrent observations. Firstly, of course, it was adopted when the scribe school was Akkadianized. More significant, however, are two other points. Firstly, the method of quadratic completion, the very trick that underlies the whole development of second-degree "algebra", was designated "the Akkadian [method]"[31]. Secondly, the choice of 10 as the standard size (obliquely reflected in the 10′ of BM 13901 N° 23) belongs with users of a decimal number system – which in the context of early second millennium Mesopotamia means Akkadians (the alternative, the Amorite nomads, are unlikely candidates)[32].

Euclid

Once we are so far, the material from *Elements* II (and Diophantos' *Arithmetica* I) can be taken into account. The *Liber mensurationum* contains no problems dealing with two squares. If we scrutinize the solutions of the first two two-square problems of BM 13901 (N^os^ 8 and 9), however, and compare them with the other two- and multi-square problems, the outcome is striking (cf. p. 11): All of these follow the general pattern we would choose by routine – N° 14, for instance, the pattern of

$$x^2+(\tfrac{2}{3}x+5')^2 = 25'25'' ,$$

and N° 18 that of

$$x^2+(x+10')^2+(x+2\cdot10')^2 = 23'20'' .$$

N^os^ 8 and 9, to the contrary, are solved by a method that cannot be generalized (as is N° 23), and which points to *Elements* II.8–10.

Propositions 1–7, on their part, are closely connected to the set of problems which we have already been able to ascribe to the early surveyors' tradition. We may hence conclude that the whole group of

[31] This is told in the Susa text TMS IX, see [Høyrup 1990: 326] – in particular note 143, which explains why the interpretation of the original text edition, widely accepted in the secondary literature, is impossible.

[32] The relation between the sides 10 and 10′ is a further argument that the subscientific tradition cannot descend from the school. 10′, as we have seen, is a unique choice in the Old Babylonian corpus; it is highly unlikely that this particular problem, with its strong though fictitious smell of practical surveying, should have been constructed around the anomalous side in question and then, by accident, have allowed normalization as 10 when surveyors, hypothetically, saved a small part of the knowledge of the school from oblivion.

Euclidean propositions is connected (as are their Diophantine cognates), not directly to the Old Babylonian school discipline as traditionally claimed by those who consider it as a geometric reinterpretation of algebraic knowledge, but to the "naive" riddles of the surveyors.

Further developments

This certainly does not exhaust the sources for the survival of the surveyors' tradition and its connections to the literate traditions. Analysis of Savasorda's *Liber embadorum*, of Leonardo Fibonacci's *Pratica geometrie*, and of the geometrical part of Luca Pacioli's *Summa de arithmetica*, shows that these authors draw, directly or indirectly, on at least three distinct works or traditions with roots in the surveyors' tradition (beyond Gerard's version of the *Liber mensurationum*, which is used by Leonardo), some of them already integrated to some extent with Euclidean methods[33]. Similarly, Abū Bakr's treatise integrates the material with *al-jabr*. This final phase is thus characterized by interactions going in all directions between the surveyors' tradition as adopted by the literate mathematical culture, and the traditions which had once been inspired by it.

Nor does the above exhaust what can be told about the development of the surveyors' tradition itself. A whole new group of problems, dealing with rectangles and their diagonals, appears to have been adopted (whether from elsewhere or by fresh development cannot be decided) at some moment between 500 B.C. and 200 B.C. They are found for the first time in a Seleucid clay tablet (BM 34 568, [ed. MKT III, 14–17]), which looks like a list precisely of *new* problems and methods: the only "classical" problem dealing with a rectangular diagonal ($A = \alpha$, $d = \beta$) is indeed omitted. All these problems recur in the *Liber mensurationum*, and again in Leonardo's *Pratica geometrie* and in Pacioli's *Summa*[34]. Their general impact on the development of algebraic thinking, however, appears to be negligible.

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[33] Space does not allow presentation of the evidence – but see [Høyrup 1993].

[34] Once more, space only allows a reference to the fuller discussion in [Høyrup 1993].

IV. Algebras?

If the surveyors' riddles have been so influential in the development of algebra (or algebras), wouldn't it be justified to regard this technique itself as an early form of algebra?

We may choose the easy way and answer by a definition. Even definitions, however, may turn out to be inadequate, and before we settle for one some general considerations will be useful.

"Algebra" nowadays designates a complex, not a single technique or a simple concept. The technique for solving equations is understood as algebra – but so is the theory of the solvability of equations, and the theory of groups etc. *Algebra* is in fact a collective name for a plurality of *algebraic ways of thought*, evidently related either logically or historically with each other, but certainly neither coinciding nor sharply to be distinguished from other ways of thought. The algebraic ways of thought may be said to constitute *today* a kind of Wittgensteinian natural family. *Today* only: if we go back in time, the various components of the family do not belong together. Thus *Elements* X, which nobody would otherwise identify as a piece of algebra, comes close to modern group theory in its classification of irrational magnitudes and determination of the relation between the classes.

Michael Mahoney [1971: 372] proposes that we distinguish between "algebra" and a more general "algebraic approach", and takes the following three characteristics to delimit modern algebra: (i) the use of a symbolism which allows us to extract "the *structure* of a problem from its non-essential content", and on which we may operate directly; (ii) the search for "relationships (usually combinatory operations) that characterize or define that structure or link it to other structures"; (iii) abstraction and absence of "any ontological commitments". To this we may add, with Viète, that (iv) algebra, if at all to be characterized as such, should be *analytic;* in fact, without this analytic character, Mahoney's criteria give no sense.

Most of what we have looked at – the surveyors' riddles, Old Babylonian school "algebra", *al-jabr* – was analytic; as far as *Elements* II is concerned, the synthetic presentation refers implicitly to an underlying analysis. None of it, on the other hand, fulfils Mahoney's three criteria to the letter, but the shortcomings in this respect differ from case to case.

The surveyors' riddles fail on all accounts. A technique that operates directly with the entities that define its problems is *eo ipso* ontologically committed, and does nothing to extract the structure of problem from "non-essential content"; the predilection for a few fixed configurations (e.g, the 10×10-square) underscores the point. The lack of systematic variation even of coefficients (not to speak of problem types) discloses a corresponding lack of interest in what defines or characterizes the structures in question.

Elements II is also ontologically committed to geometry – only later in the work, when investigating "magnitudes" in general (in particular in book V, the theory of proportions) is Euclid leaving ontology behind. But the heart of the "critical" undertaking is the question *why* the techniques work, and thus implicitly an attempt to separate the essential structure from non-essentials (however much this separation is hampered by the ontological allegiance – the more general theory of the "application of areas" in book VI, however, can be seen as a further effort to eliminate non-essentials).

The "critical" approach of *Elements* II is a symptom, first of all, of the "scientific" character of Greek mathematics, and thus not so much of "algebraization"[35]. Babylonian school "algebra" – "critical" only to a modest extent – followed a different path. Most of the texts deal with geometrical entities, it is true, which might be taken as an expression of ontological commitment. Modern school teaching of algebra is no different, however: its *x*'s and *y*'s mostly stand for pure numbers (scarcely more abstract than the ideal geometrical plane). Geometry and numbers thus make up the *basic conceptualizations* of Old Babylonian "algebra" and modern school algebra, respectively; but in both cases, the basic entities may *represent* entities of different ontological character. By using geometry as *representation*, the Babylonian technique shows itself to be *functionally*

[35] The parallel aspect of Mahoney's criteria shows, correspondingly, that modern algebra, and indeed algebra since Viète, resulted when "scientific" mathematics took possession of the algebraic technique of the *maestri d'abbaco* and *Rechenmeister*.

abstract, *functionally* devoid of ontological commitment[36].

The systematic variation of problems, a distinctive characteristic of the Old Babylonian school, was certainly not meant as a search for underlying structures; to some extent it reflects an interest in probing the tools of the profession, but its main purpose was probably that of training (not least, training the use of the sexagesimal number system). None the less – the surviving texts leave no doubt about this – it led to at least intuitive insight into formal structures and relationships.

In many respects, the case of *al-jabr* is similar. Even here, critique and the understanding of formal structures and relationships were not primary aims but still the outcome at least at the intuitive level. *Al-jabr*, even more explicitly than the Old Babylonian technique, was used (from the beginning of the tradition as we know it) for *representation*, and its "possessions" and "roots" were certainly functional abstracts. The question of the basic conceptualization, however, is more complex. Fundamentally, the "possession" is an amount of money; yet already in al-Khwārizmī's treatise it has come to be primarily *a number*. Later authors, moreover, under the impact of the geometrical demonstrations, tended to use this geometry as their basic conceptualization.

All three literate traditions thus agree with some of the criteria which characterize the modern algebraic mode of thought ("algebra" *tout court*) but not with all. Whether they are *algebras* can thus only de decided by *fiat*, by definition. Since the Euclidean approach is primarily related to modern algebraic though by being critical, i.e., in so far as it shares the general character of Greek mathematics, it seems reasonable *not* to regard it as algebra; the uses to which Apollonios puts the technique of application of areas – uses which induced Zeuthen [1886] to characterize it as "geometric algebra" – should rather make us see it as *a substitute for algebra*.

On the other hand, since Modern (and ultimately modern) algebra arose when the critical approach was imposed upon the Renaissance descendant of *al-jabr* (which was not significantly different from the discipline of al-

[36] Fifty years' misreading of the geometrical texts as dealing with nothing but numbers and arithmetical operations shows (it may be added) that the ontological commitment to geometry cannot be strong; the structural features of the text corpus that exclude the numerical interpretation are obviously not too conspicuous.

Khwārizmī and Abū Kāmil), it seems sensible to regard *al-jabr* as a pre-Modern, pre-critical algebra (if we do not, Cardano's *Ars magna* is no book about algebra). Finally, since the Old Babylonian technique agreed with our criteria grossly to the same extent as did *al-jabr*, even this will have to be accepted as a pre-critical algebra.

The surveyors' tradition of riddles about measurable lines and areas, itself no algebra according to any reasonable delimitation, was thus the parent of one algebra (the Old Babylonian version), an important ingredient in the transformation of the subscientific *al-jabr* technique into another algebra, and finally the inspiration for the first critical investigation of algebraic patterns of thought and indeed for the creation of a substitute for algebra. Quite an impressive score for a tradition whose very existence all its literate legatees and debtors leave unmentioned.

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