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Faggruppen for filosofi og videnskabsteori

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Section for philosophy and science studies

HOW TO EDUCATE A KAPO
or
Reflections on the Absence of a Culture of Mathematical Problems in Ur III

JENS HØYRUP

FILOSOFI OG VIDENSKABSTEORI PÅ ROSKILDE UNIVERSITETSCENTER

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To Bob (Englund) and Hans (Neumann)

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Personal prehistories

Firstly:

Some fifteen years ago, when Robert Englund had recently changed his dissertation theme from "Ur III-Fischerei" to "Organisation und Verwaltung der Ur III-Fischerei", he told me (halfway jestingly) to be alarmed, discovering himself to have become a Stalinist: namely because analysis of the sources obliged him to conclude that Ur III was a slave society. This agreed better than he liked with the orthodox Stalinist version of historical materialism, according to which "slave society" follows after "primitive communism", without any intervening "Asiatic mode of production".

I suggested he should not worry too much. In the present case, the orthodoxy had little to do with the contents or style of Stalinist policies, apart from its being *made* an orthodoxy. Stalin had just happened to listen to the best authority on the question he could find: V. V. Struve, who, when becoming curator of the Hermitage cuneiform collection, had started reading Ur III accounts and had been led to the same conclusion as later Englund.[1]

Further on, Englund sharpened his views considerably: in view of the strict regime to which not only the workers but also the overseer-scribes were submitted he would now speak of a "*Kapo* economy" – a *Kapo* being (in case anybody should not have heard about the system) the KZ prisoner responsible for the work or organization of a group of fellow prisoners, in constant danger of being reduced to the status of an ordinary prisoner as soon as he did not fulfil his tasks in a way that contented the SS. And, as formulated in the concluding words of the published version of the dissertation [1990: 316], the understanding of working conditions conveyed by the administrative texts

> kann vielleicht helfen, sich in den historischen Darstellungen des 3. Jahrtausends v. Chr. die Kosten der babylonischen Paläste und Statuen plastischer vorzustellen.

In a volume published in the same year[2] we find a more detailed formulation in a commentary to the accumulated deficit in the yearly balance of

[1] See [Diakonoff 1969: 5].

[2] [Nissen, Damerow & Englund 1990: 89, cf. *id.* 1994: 54]] – this chapter signed by Peter Damerow and Robert Englund.

an overseer-scribe:

> Aus anderen Texten wissen wir, welche ernsten Konsequenzen solch eine lückenlose Überwachung der anwachsenden Fehlbeträge für den Aufseher und seinen Haushalt mit sich brachte. Die Fehlbeträge mußten offenbar um jeden Preis beglichen werden. Verstarb ein Aufseher, so wurde sein Nachlaß herangezogen, die Schuld zu Tilgen. Das bedeutete in der Regel, daß die verbleibenden Haushaltsmitglieder selbst in die staatlichen Arbeitertrupps eingegliedert wurden, die die von den Aufsehern überwachten Arbeiten zu verrichten hatten.
>
> Das waren die Arbeitsbedingungen der Aufseher, über die in den Verwaltungstexten allein Buch geführt wurde. Über das Schicksal der Arbeiterinnen und Arbeiter sind dagegen kaum Informationen überliefert. Sogenannte "Musterungstexte" berichten regelmäßig über in großer Zahl entflohene Arbeiter. Man kann sich angesichts der totalen Überwachung aller Leistungen der Trupps, in die sie eingegliedert waren, die Gründe leicht ausmalen.

Secondly:

Already before I started my conceptual analysis of the Old Babylonian "algebraic" texts in 1982 I suspected these to represent a *new* genre, irrespective of current opinions. In [1980: 20–22] I had thus written that[3]

> the influence of the school on Sumerian mathematics was (as far as we know it from published material[4]) restricted to the *systematization of applied mathematics*. [...] Sumerian mathematical texts are concerned with *real* "real-world-problems"; this does *not* imply that they are always realistic: One school text from the Sargonic epoch[5] deals with a field as long as 1297.444 km (given to that precision [...]). [...] In historical retrospect, this characteristic is typical of the teaching of even

[3] I change the numbering of notes and the format for the references but leave my original text intact (though truncated, and with omission of a number of notes) in other respects.

Evidently, I would now formulate much of what is said in different terms (not least avoiding the notion of an Old Babylonian "pure mathematics"). To the extent one can distinguish substance from formulation, I believe most substantial points remain valid.

[4] [...] comparison with the distribution of literary texts seem to indicate that Sumerian precursors of later Babylonian pure mathematics are really non-existent: Indeed, even if most literary texts are known only from later versions although they have Sumerian or Sargonic origins, a reasonable number of literary tablets are known illustrating the tradition all the way back to 2500 B.C. [Hallo 1976, *passim*; Alster 1974: 7]. *No* such precursors for Babylonian pure mathematics exist.

[5] See [Powell 1976: 428*f*].

practically oriented mathematics when this teaching has been *institutionalized* and thereby has become the task of a partly closed milieu. [...].

The practical even if sometimes abstract character of Sumerian mathematics is in perfect harmony with what little we know about the curriculum of the Ur III school: It was purely utilitarian, and had no room for *l'art pour l'art*[6].

Babylonian culmination

The practical fixation of Sumerian mathematics and mathematics teaching may perhaps be regarded as a consequence of the integration of the Sumerian school and the scribal profession in the state administration. We may guess that the unequivocal public attachment of the scribal function may have restricted the ideological autonomy of the scribal school and thereby its institutional independence.

This is only speculation, and we may leave it as it stands. [...].

Individualism in the Old Babylonian society was not confined to the commercial sphere. [...].

In this situation, the scribal profession seems to have become more independent as a social body; at least, it became less unequivocally attached to the public authority and function. [...]. At the same time[7], a genuine pure mathematics was developed [...], based in part on *methods* with no relevance for down-to-earth practical tasks – derived, truly, from practitioners' methods, but transformed and developed by the contact with the theoretically generated problems.

As I engaged afterwards in my close reading of the "algebraic" texts, I found confirmation in the predominance of Akkadian terminology; in the Akkadian language structure even of almost exclusively logographic texts and, not

[6] [Cross reference to an earlier note, in which is found:]
According to hymns made in the name of king Šulgi, the curriculum of the Ur III school contained writing, arithmetic, accounting, field measuring, agriculture, construction, and a few subjects the names of which are not understood; cf. [Sjöberg 1976: 173*f*].

[7] The precise chronology of the process is at least for the moment not to be known. Still, the decisive steps must be placed in the earlier part of the period, since a number of characteristic texts dating from c. 1800 B.C. have been found in Tell Harmal [...].

A limit *post quem* is obtained from the observation that the problem type most characteristic of the new pure mathematics – the "equation" of the second degree – is intertwined with the use of the full potentiality of the sexagesimal number system; it seems [...] that this type of mathematics can only have been developed after the use of sexagesimals had been generalized.

[...] The language was Akkadian, the language of the new literary creativity.

least, in the fact that the only terms that occur exclusively or almost exclusively in Sumerian (or as loanwords provided with Akkadian declination) are those that belong unambiguously to the sphere of practical computation: uš, sağ, a.šà, igi, igi.gub, íb/ba.si$_8$.

Late in the 1980s I got the hunch – by then still built on very tenuous evidence – that a tradition of surveyors' recreational riddles might be "older than – perhaps even a source for – Old Babylonian scribe school 'algebra'" [Høyrup 1990a: 275], but invested no more than this single line in the hypothesis.[8]

I had no suspicion by then that the various strands of what precedes might end up being intertwined. That they are only struck me at my third or fourth return to the general structure of the Old Babylonian mathematical vocabulary in 1998. The present paper is meant to tell that story.

The Old Babylonian mathematical operations and their vocabulary

The (more or less) technical terms used in Old Babylonian mathematical texts fall in two main groups: (1) terms for operations, and (2) the meta-language which allows the formulation of problems and the higher-level explanation of the procedures by which they are solved.

The former group can be subdivided thus:

Additive operations

Two operations belong to this group. One is the "identity-conserving" addition in which an extra piece is joined to a quantity; it is always concretely meaningful. It is designated by the Akkadian term *waṣābum*, with the Sumerogram daḫ, for both of which I shall use the standard translation "to append".[9] In agreement with the "identity-conserving" character of the

[8] The manuscript for the publication in question was finished in 1987; in [Høyrup 1990b: 79*f*], written in 1989, the argument is elaborated and some supplementary evidence is cited.

[9] For convenience, I make use of a system of fixed "standard translations" in the following: it is easier to insert the conjugated forms of "append" in an English phrase than those of *waṣābum*; for non-Assyriologist readers it may also be easier to connect them to the common root. The ones I use here differ slightly from the set used in my [1990a] but coincide with those of [Høyrup, forthcoming/a].

For the reasons that lead to the interpretation of the terms I shall only refer to operation, no particular term for the corresponding "sum" appears to exist.

The other operation is symmetric, and does not presuppose the addition to be concretely meaningful. It heaps or "accumulates" (the measuring numbers of) two or more addends, connecting them with the word *u* ("and"), and may thus be regarded as a genuine arithmetical operation. The main Akkadian term is *kamārum*, to which correspond the Sumerian term g̃ar.g̃ar and the unexplained logogram UL.GAR. The corresponding sum has a name: *kumurrûm*, with logograms g̃ar.g̃ar and UL.GAR. Occasionally, other nouns derived from *kamārum* are used for the sum, most remarkable among which is the plural *kimrātum*, apparently a reference to the sum as consisting of still identifiable constituents. On some occasions, *u* alone serves in the same function; in one text (BM 85200+VAT 6599) an abbreviated form of the term used for the sum total in accounting (NIGIN) turns up twice; in both cases, two *numbers* – *viz* a pair belonging together in the table of reciprocals – are added.

Subtractive operations

Two "subtractions" occur in the texts, *removal* and *comparison*. Both are always concrete.

Removal may be seen as the inverse of appending. The main term is *nasāḫum*, "to tear out", with Sumerogram zi. It can only be used when the subtrahend is part of the entity from which it is subtracted. Most texts from early eighteenth-century Eshnunna and some from Goetze's equally early "group 1"[10] prefer the Akkadian *ḫarasum*, "to cut off", with no proper Sumerogram;[11] in contrast, zi is used frequently even in predominantly syllabic texts.[12]

Comparison indicates how much one magnitude *A* exceeds another

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my preceding publications, in particular to [Høyrup 1990a].

[10] Goetze, in [MCT, 146–151]]. For the chronology of the group, see [Høyrup 2000a: 149]; when speaking in what follows of text groups I refer to the new delimitations of Goetze's original groups established in this latter publication.

[11] kud, used in the late Old Babylonian TMS XXVI, may stand for *ḫaraṣum* but also for *nakāsum*, "to cut down", or *haṣābum*, "to break off" – or, not to be excluded, for *nasāḫum*.

[12] For the use of *tabālum*, "to withdraw", and *šutbûm*, "to make leave", for specific removals, see [Høyrup 1993].

magnitude *B which it does not contain*. It is no inverse of *kamārum*, and cannot be the reversal of any addition (since the sum always contains the addends).[13] The phrase in use states that *A eli B d itter/īter*, "*A* over *B*, *d* it goes/went beyond" (from *eli ... watārum*, "go beyond", "be(come)/make greater than"), with the Sumerographic equivalent *A* ugu *B d* dirig.[14]

"Multiplications"

Four different operations can in some way be understood as "multiplications".

One is the multiplication of number by number as found in the tables of multiplication, to which corresponds the Sumerogram a.rá (from RÁ,[15] "to go") and the phrase *a* a.rá *b*, meaning "*a* steps of *b*". The term has no Akkadian counterpart but gives rise to the loanword *arûm*<**ara-um*.[16]

The second is the determination of a concrete magnitude by means of a multiplicative operation. It is used in multiplications by technical constants and metrological conversions; it serves when volumes are determined from base and height and in the calculation of areas that are not implied by the construction of a rectangle (that is, areas of triangles, of trapezia and trapezoids, and of rectangles which *are already there*). The core term is *našûm*, "to raise", with the Sumerogram íl. Another Sumerogram used in the same

[13] Because of the symmetric character of accumulation, its actual inverse is the *splitting into* or *singling out of* components (*bêrum*, with no Sumerographic counterpart in the mathematical texts). It may be no accident that the term is used in AO 8862, the very text that speaks of the accumulation as a plural *kimrātum*.

[14] The relatively rare comparison made the other way round, the statement of how much *B* falls short of *A* (using the verb *maṭûm*, "to be(come) small(er)", with Sumerogram lal), is discussed in [Høyrup 1993].

[15] The verb takes on a number of different forms depending on grammatical number and aspect [SLa, §268]: g̃en and du (singular perfective and durative), re₇ and su₈.b (plural ditto). Since both g̃en and du are written DU = RÁ, it is convenient to write the verb as RÁ in order to keep present the relation with the term a.rá (the pronunciation of which is certified by the loanword *arûm*).

[16] Admittedly, the metaphor of "repeated going" expressed in Akkadian (*alākum*) and used in a general way (for multiplication as well as repeated "appending") is found in various Old Babylonian texts from Susa. *The pattern of thought* behind the term a.rá thus found expression in Akkadian; but it did not produce an Akkadian equivalent of *the term*.

function is nim, logographically connected to *elûm* and *šaqûm* and their various derivations (both "to be/become/make high")".[17]

The third multiplicative operation is the repetition of a concrete magnitude an integer number of times ("until *n*", $2 \leq n \leq 9$). It is spoken of in Akkadian as *eṣēpum*, "to double", with the Sumerographic equivalent tab (whose original sematic range is wider, for which reason Seleucid mathematical texts could readopt it as a logogram for the identity-conserving addition).

The fourth operation is better dealt with under the following heading.

Rectangularization, squaring and "square root"

This operation, indeed, consists primarily in the "building" (*banûm*) of a rectangle and is only a multiplication in so far as the computation of the appurtenant area is treated as inherent in or implied by the construction. The central term is *šutakūlum*, "to make [two segments *a* and *b*] hold each other" (*viz* as sides of a rectangle – at times grammatical constructions are used which rather imply that *a* together with *b* contain or "hold" the rectangle). ì.gu_7.gu_7 (from gu_7, "to eat") is used as a logogram, probably because of the phonetic near-identity between *šutākulum*, "to make eat each other", and *šutakūlum*.[18]

[17] The original mathematical use of the term is connected to the determination of volumes. In these, indeed, the base is invariably "raised" to the height; in all other cases, the order of the factors is random from a mathematical point of view, and depends first of all on stylistic criteria – as a rule, it is the quantity that has been computed in the preceding operation that is "raised" to the other factor, irrespective of its meaning or role in the computation – cf. [Høyrup 1992: 351*f*].

As is well known, volumes were measured in area units, which implies that these were thought of as provided with a virtual standard thickness of one kùš (cubit). In consequence, determination of a prismatic volume meant that this virtual thickness was "raised" to the real height. Raising multiplications are thus operations of proportionality, and may hence be said to be "category-conserving".

[18] Traditionally, most workers (Thureau-Dangin being the chief exception), have taken the logogram as an argument that the term should be read *šutākulum*, "to make [*a* and *b*] eat each other", which fits the cuneiform orthography just as well. However, certain texts refer to a segment that has been submitted to the operation with the relative clause "which you have made hold/eat", while others use a noun *takīltum* with precisely the same function; the latter term can only derive from *kullum* and mean "which is made hold". Of course, rebus- or pun-like substitutions like that of *šutākulum* for *šutakūlum* constitute the very fundament for the cuneiform writing

Beyond i.gu_7.gu_7 and the abbreviation i.gu_7, several other logograms are used: UL.UL (probably to be read du_7.du_7, for *nitkupum*, "to butt each other"); UR.UR; and NIGIN, which can be interpreted as a contracted LAGAB.LAGAB. In all four cases, the reduplication is probably not to be understood as a genuine Sumerian grammatical form but rather as a way to render in pseudo-Sumerian the reciprocity of the process that the Akkadian language presents by means of the Št and Gt-stems (*šutakūlum* and *nitkupum*, respectively).

Several texts play with repeated numerical multiplication, but no specific term for powers *of numbers* exists. *Squaring*, as a specific process, is *geometric* squaring.

The Akkadian term for the square configuration is *mitḫartum*, a term which refers to a confrontation of equals (*viz*, equal sides), and which when expressed as a number coincides with the side of the square.[19] The word derives from the verb *maḫārum*, whose (causative-reflexive) Št-stem *šutamḫurum*, "to make *s* confront itself", designates the construction of a square with side *s*. Often, however, one of the terms for rectangularization is used instead. With respect to one side, the other side meeting it in a common corner is considered its *meḫrum* or "counterpart", sometimes replaced by the Sumerogram gaba.[20]

Some texts use LAGAB and NIGIN as ideograms (not necessarily logograms, the sign LAGAB *being* a square) in the same functions as *šutamḫurum* and/or

system and should not surprise us.

[19] Its primary reference is thus the square frame, parametrized by the side, not as with Euclid that (area) which is contained by the boundary.

[20] *meḫrum* is derived from the same verb as *mitḫartum* etc. gaba, on its part, is wholly independent, meaning rather "identical copy". Anybody who has tried to translate a technical terminology from one language into another will have observed that etymologically related terms often end up having unrelated translations, whereas it is a rare luck to find etymologically related translations for original terms which, though semantically related are etymologically unrelated. Anticipating what is to follow we may conclude that the *mitḫartum*/*šutamḫurum*/*meḫrum*-terminology is likely to have originated in Akkadian.

mithartum. Some series texts use íb.si$_8$ in the same function;[21] however, the normal function of this important term is different.

Originally it is a Sumerian finite verb form, and it often occurs as a verb in phrases "Q.e *s* íb.si$_8$", where *Q* and *s* are numbers, $s = \sqrt{Q}$. /.e/ is the "ergative" or "locative-terminative" suffix", si$_8$ a verb stem meaning "being equal"; /íb./ combines a mark of finiteness /ì/ with the "inanimate pronominal element" /b/ – *s*, indeed, is no person. In total, the phrase therefore has to be translated "by *Q*, *s* is equal".[22] The meaning is that when the area *Q* is laid out as a square, it is flanked by *s* as side of this square (equal of course to the other sides).

Other texts have left behind the etymology, and use íb.si$_8$ as a noun, the name for this side (with Jöran Friberg we may call it "the equalside") – so to speak the geometrical equivalent of a square root.[23]

Quite a few of the texts that use the term as a noun employ the homophonic (unorthographic) writing íb.si. Other texts use ba.si or ba.si$_8$ (the latter form also occurs as a verb); the shift si$_8$>si simply shows that the etymology is no longer thought of, but the pronunciation still Sumerian;[24] /ba/ is an alternative prefix which probably contains a locative

[21] A single procedure text, moreover (YBC 6504), uses íb.si$_8$ in parallel with du$_7$.du$_7$, in a construction where it might function logographically for *šutamhurum*. In BM 15285, a catalogue text concerned with the subdivision of squares into other geometric figures, it alternates with *mithartum* as a term for the square configuration.

[22] In earlier publications I have blindly accepted the interpretation of .e as an ergative suffix, which would give the phrase the meaning "(caused) by *Q*, *s* is made equal". This was originally proposed by Thureau-Dangin, and was indeed the only possibility within the arithmetical reading – *the number* 9 cannot meaningfully be claimed to be close to *the number* 81.

[23] Even when no declination elements allow us to distinguish cases, the word order shows whether a verb or a noun is meant, both Sumerian and Akkadian being verb-final. en.nam íb.si$_8$ thus means "what is equal?", whereas íb.si$_8$ en.nam means "the equalside, what?".

[24] Thureau-Dangin, in contrast, believed íb.si$_8$ to be a logogram that was pronounced *mithartum*; in a few cases (*viz* when the reference *is* the square configuration) this was certainly the case, but mostly not.

element /a/, indicating that the process takes place "(out) there".[25] At times, the noun appears as a loanword *basûm*, further proof that the pronunciation remained Sumerian.[26]

The function of the íb/ba.si$_8$ is not restricted to the case of squares. It may also refer to the side of a cube, which should not disturb us – even the sides of a cube are equal and close to the volume they contain. But true generalizations also occur, somewhat similar to our reference to the "root(s)" of an equation, derived in tangled ways from the concept of a square root.

Division, parts and igi

Division, as we use the word, is both *a problem* – to solve the equation $bx = a$ – and an operation. The familiar assertion that division does not exist in Babylonian mathematics refers to the absence of division as an arithmetical operation of its own.

As is well known, the way a division *problem* is dealt with depends on whether the divisor b is regular or not, that is, whether its reciprocal has a finite expression in the sexagesimal system. The reciprocal of such a number n is called igi n ĝál.bi ("[of 1,] its igi n ĝál"[27]), often shortened to igi n ĝál or igi n. When a division by such a number n is to be

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[25] The prefix /ba./ is often used regularly with the identity-conserving subtraction zi, "to tear out"; in contrast, the identity-conserving addition daḫ, "to append", is often preceded by /bi./, which suggests a "here", a closer contact.

In the texts from early eighteenth-century Eshnunna, some texts use the ba- and some the íb-form for the quadratic case (the ba-form always as a noun). ba.si$_8$.e is also used in the quadratic case (but as a verb) in two atypical texts from early Old Babylonian Ur (UET V, 859 and 864). Elsewhere, the ba-form is only used for the cubic and for generalized cases (cf. imminently).

[26] Some publications use the transliteration íb.sá instead of íb.si$_8$. Later lexical lists, indeed, render the pronunciation of the Sumerian verb as *sa-a*; the preference of most editions of mathematical texts for si$_8$ is supported by the occasional homophonic shifts to íb.si and ib.sí, which however might as well be íb.se and ib.sé. Since we also find the syllabic writing ba.se.e (which can *not* be ba.si.i) for *ba.si$_8$* (IM 52301 rev. 7, 9), the real phonetic value might be between *-se* and *-sä*.

[27] Strictly speaking, early tables of reciprocals show that the meaning is "[of 1`,] its igi $n$ gal". These tables, indeed, list $^2/_3$, $^1/_2$, $^1/_3$, $^1/_4$, etc. of *sixty* = 1` – see Steinkeller [1979: 187]. In later times, genuine reciprocals in our sense may have been thought of.

performed, the texts ask for igi n to be "detached" (*patārum*/du$_8$), after which the dividend is "raised to" the igi. In modern terms, division by n is thus performed as multiplication by $^1/_n$. A couple of mathematical texts from Eshnunna replace igi with *pa-ni*, "in front of",[28] which suggests a reference to the table of reciprocals, where igi n is indeed present "in front of" n – a folk etymology close at hand, not least because g̃ál means "to be/place (somewhere)", "to be at disposition". However, since texts from Lagaš from c. 2400 BCE speak of $^1/_3$, $^1/_4$ and $^1/_6$ as igi n g̃ál, it is clearly nothing but a secondary folk etymology, the opinion of E. M. Bruins ([1971: 240] and elsewhere) notwithstanding. We also find an Akkadian loanword *igûm*.

If the divisor is not (or cannot easily be seen to be) regular (a recurrent situation in mathematical procedure texts though hardly in practical computation, all relevant technical coefficients being chosen to be regular), the text takes note of the non-existence of the igi and then formulates the division as a problem – "what may I posit (*šakānum*/g̃ar) to b which gives (*nadānum*/sum) me a" – and states the answer immediately, "posit p; raise p to b, a it gives to you" (or some slight variation on this pattern). Mathematical problems being constructed backwards from the solution, this could always be done without difficulty.

Bisection

When dividing by a number which, so to speak, is 2 only "by accident", our texts find "igi 2" = 30′ and "raises" to that number. Beyond that, a particular sign and a corresponding set of words (*mišlum*/šu.ri.a) for the half exist. They are used, for instance when the width of a rectangle is said to be half its length; if one entity is said to exceed another by its half; or to indicate a measure ("half a barleycorn"). Such relations and measures are "accidental": they might as well have been slightly different.

Besides this accidental half, however, a different, "necessary" half (invariably half *of something*, never a number – namely of something which naturally or the nature of the case falls in two halves) is used in the texts,

[28] For instance Haddad 104. The lexical form *pa-nu* occurs in a text from Sippar (BM 96957+VAT 6598, in [Robson 1999: 231]) – but as an interlinear gloss, which suggests that it may have been intended as an indication of pronunciation and nothing more.

the *bāmtum* (no Sumerogram seems to exist[29]). It occurs in places where something is bisected into two necessarily equal parts: for instance when the base of a triangle is bisected for the purpose of an area calculation; when the same is done to the sum of opposing sides in a trapezium; and when the radius is found from a circular diameter.

The "necessary half" is invariably found by "breaking" (*ḫepûm*/gaz). This verb, on the other hand, has no other function in the mathematical texts; it always goes together with *bâmtum* (or with ½ or šu.ri.a in the rare texts where these are used logographically for *bâmtum*). In the mathematical texts (but only here), "breaking" is thus the same as *bisection*.

Structure and metalanguage

These mathematical operations are used within texts that are organized with a particular higher-order structure – enunciation and exposition of the procedure, hypothetical-deductive arrangements, relation between "true" entity and representative – and which in order to express that structure make use of what we may call a metalanguage – names for unknown quantities, terms that indicate equality, terms that delimit expressions (corresponding to the parentheses of our algebraic expressions), terms that announce results, etc. Of importance for the present argument are the following categories.

Structuration

The mathematical procedure texts are arranged into statement and prescription, with a corresponding distribution of grammatical person and tense. At face value, the structure corresponds to the scene of the scribe school

[29] Non-presence of a term in the extant text material is in general a weak argument. But absence from texts where it *should* have occurred is strong. Such texts are Str 366, 367, 368, all of which insist on writing everything with Sumerograms (except grammatical particles that do not exist in Sumerian), although in clearly Akkadian sentence structures: they eschew the word (Str 366), or they use the number sign ½. This is also done in VAT 7532 and VAT 7535. The text YBC 6504 (already mentioned for its unique use of ib.si$_8$ in the function of *šutamḫurum*, "to make confront itself") has recourse to šu.ri.a.

BA and BA.A, used in the function of *bāmtum*, are taken in [MKT] to be Sumerograms; actually, we are confronted with elliptic writings or, more likely, with irregular assimilations to the pronominal suffix *-šu*, *bâššu* < **bāmat-šu*; the noun **bûm* [MCT, 161], constructed backwards from similar forms, should also be *bāmtum*.

as known from other sources: the master states a problem ("I have done so and so"), next the instructor or "big brother" explains in the present tense or the imperative what "you" (the student) should do, referring occasionally to what was said by "him" (the master). A group of texts from Eshnunna from the early eighteenth century BCE, however, start by asking "If somebody has asked you thus, 'I have done so and so'", suggesting instead that the format belongs originally with mathematical riddles. Early texts from the south, on the other hand – both many of those which belong to group 1[30] and those from early Old Babylonian Ur – deviate from the pattern that prevails elsewhere. All in all, the format thus seems not to have originated as a portrait of the scribe school but to have been imported into it; below (text around note 48) this conclusion will find further confirmation.

Within this format, the logical structure of the texts and the way to compose fairly unambiguous mathematical expressions is also standardized, though not uniformly in all text groups.

Occasionally, the hypothetical-deductive structure of the problem is made explicit by an introductory *šumma*, "if" – also familiar as the opening of the protases of omina ("if the liver looked so and so", etc.). Since it is found in most of those texts from Eshnunna that do not carry the full "if somebody has asked you thus" (and in this function almost exclusively in texts stemming from the same northern part of Babylonia), the presence of the term in this place may originally have been a vestige of that phrase. In one text from Eshnunna (Haddad 104), however, as in certain other texts, *šumma* is used to introduce variations of an exemplar, carrying thus the meaning "if (instead the situation is as follows:)". *šumma* may also serve to introduce a smaller piece of deductive reason inside the prescription from already established foundations ("if (as you have now established) ..."), or to open the proof; it can be found in either function both in texts from Eshnunna and other localities in the periphery and in texts from the former Sumerian heartland.[31]

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[30] Thus AO 6770, AO 8862, YBC 7997, YBC 9856, YBC 9874.

[31] A division of the text corpus into "northern" and "southern" types was originally undertaken by Goetze, who based himself on orthographic criteria (almost all mathematical texts known in 1945 came from illegal diggings and were thus of unknown origin). In my [2000a] the inclusion of the text groups from Eshnunna and

inūma, "as", is used in a couple of texts from the periphery to mark a piece of deductive reasoning on already established foundations. *aššum*, "since", has the same function in some texts from the periphery and in some from the core (in the periphery specimens it mainly serves as the opening phrase of the procedure prescription).[32]

It may also be used in connection with quotations from the statement which serve to justify the pertinency of single steps in the procedure. The whole quotation is contained in the phrase *aššum ... qabûku/iqbû*, "since it is said to you"/"since he has said".

In certain cases, the statement falls into two sections, the first of which contains general information – the value of a technical constant to be used, the rent to be paid per bùr of a field, etc. – whereas the second presents the actual problem. The second section may then be introduced by the word *inanna*, "now".

Even the prescription may be divided explicitly into subsections. Such divisions may be marked by the verbs *saḫārum* "to turn around", *târum*, "to turn back", or nigin(.na), found in the late group 6 (from Sippar) and probably meant as a logogram for *târum* (this term occurs in related texts but *saḫārum* not). At least in the text AO 8862, *saḫārum* appears to be used in the statement about a quite concrete walk around a field which has just been marked out, and *târum* about a return to the starting point. *târum* is used in a similar way in texts from several text groups; it seems no unreasonable guess that this concrete application may have been the origin of the abstract usage as a textual delimiter in the prescriptions.

Many problems – particularly those of "algebraic" character – are shaped as *equations*, statements that (the measure of) a more or less complex quantity equals a number; also present though less common are equations that declare that the (measure of) one quantity equals (the measure of) another quantity. In the former case, the equality is mostly implied by the enclitic particle *-ma* on the verb. In the latter case, the term *kīma*, "as much as", may be found; in the series text but nowhere else it is written logographically as

Susa led me to replace Goetze's division by a distinction between the (former) "Sumerian core" and its "periphery".

[32] Not, however, in texts beginning "if somebody ...".

gin$_7$.(nam), the Sumerian equative suffix.

A term for equality that may serve as a kind of bracket when complex quantities are constructed verbally is *mala*, "so much as". It is found in the expression "so much as *a* over *b* goes beyond", meaning (a–b). In the series texts it is replaced by the Sumerian interrogative pronoun a.na.

The numerical value of a quantity Q may be asked for in two ways, either by the question *mīnûm Q*, "what (is) Q", with logographic equivalent en.nam, or by the question *Q kī masi*, "Q corresponding to what?".[33] Collective questions for each of several values may be asked for with the question *kiyā*, "how much each" .

"Variables"

What permits us to speak of an Old Babylonian "algebra" is the existence of a *standard representation*, a mathematically structured domain onto which problems dealing with entities belonging to other domains but entering in mathematical relations that are homo- or isomorphic with those of the standard domain may be mapped and then solved by analytic procedures (implying that in practice the standard domain is reduced to its mathematical structure, and that it is thus functionally abstract). In the Old Babylonian case, the standard domain is that of squares and rectangles[34] with measured or measurable sides and corresponding areas. In a number of problems, these sides represent prices (actually inverse prices, quantity per monetary unit), workers and working days, numbers, or even areas or volumes;[35]

[33] In the division question, the accusative *mīnâm* is used, "what may I posit to ...". Similarly when a geometric square root is asked for by the verbal construction, "by Q, what is equal". When the "equalside" is regarded as a noun, it may be asked for by the nominative *mīnûm*, or the student is asked to make it come up", *A basâšu šuli*, or to "take" it (*laqûm*).

[34] Certain problems dealing with prismatic excavations are of the third degree. However, as their constituent elements never serve to represent entities of other kinds, they do not belong to the standard domain; instead, those excavation problems which are of the second or first degree are *represented* themselves within the standard representation.

[35] A distinction between "true width" and "width" in the Susa text TMS XVI could also mean that the width 20 of a real field (*viz* 20 nindan) is represented by a standard-domain width 20 (*viz* 20′ nindan, fit for the dimensions of the school yard). But this remains a hypothetical interpretation of a passage which seems not to be

as in more recent school algebra, however, most extant problems deal with the basic representation itself.

The names of the entities belonging to the standard domain are, for rectangles, uš ("length"), sağ ("width"), a.šà ("surface"), and for squares *mitḫartum* ("equalside") and a.šà ("surface").

Outside their function as standard representation, uš and sağ are logograms which may be replaced by the syllabic equivalents *šiddum* and *pūtum* (and, in other connections, by other words – sağ thus by *rēšum*, "head"). With exceedingly few exceptions, this does not happen when they serve within this function. Similarly, a.šà is never replaced by the syllabic *eqlum*; in this case, however, the logogram may be provided with Akkadian phonetic elements.

In the case of square configurations and sides, the situation is different (their areas are still a.šà). The configuration *may* be referred to as LAGAB or íb.si$_8$, but *mitḫartum* occurs quite as often. As to the value of the side itself it may also be spoken of as *mitḫartum*; as LAGAB provided with a phonetic complement that identifies the Akkadian pronunciation; or it may said that *s* "confronts itself" (*imtaḫḫar*).[36] A set of curiously related catalogue texts from Eshnunna, Nippur and Susa[37] refer to the side of the configuration as uš, but with a phonetic complement that identifies it as *pūtum*. Three texts[38] in contrast, speak of the sağ or *pāt*, plural "widths", of square configurations, but at least the first two may be argued to speak of "real", not standard fields.

In modern letter-based algebraic computations it is often mnemotechnically convenient and therefore customary to label derived variables which somehow fulfil the same function as the initial ones by some kind of marking – $\bar{x}$ for x, etc. The Babylonians used several similar tricks –

explainable in any other way. In general, there is no positive evidence that the distinction between "standard-domain" and "real" fields, fairly well-respected in the practice revealed by the texts, was also formulated as a principle.

[36] Or, instead, asked "how much, each, confronts itself", with the particular interrogative particle *kiyā*.

[37] IM 52916, IM 52685+52304, CBS 154+921, TMS V, TMS VI.

[38] UET 864, from early Old Babylonian Ur; BM 13901; and CBS 19761, from Nippur.

distinguishing for instance between entity and "true" (*kīnum*/gi.na) entity or between entity and "false" (*sarrum*/lul) entity, or referring to the way the new entity is produced. Even within groups of closely related texts, however, no uniform pattern for doing so can be identified, nor are "true" and "false" used exclusively in this function within the texts. All devices of the kind were clearly as much *ad hoc* as the picking of X, ξ, *u* or $\bar{x}$ for "new *x*".

Recording

Numbers occur as data in the statement, as intermediate outcome of calculations, and as final results. Several terms and phrases may be used in this connection.

Of particular importance is the verb *šakānum*, "to posit", with logogram g̃ar. The term appears to designate various kinds of material recording – "putting down" in a computational scheme, writing the value of a length or an area into a diagram, etc. Its is mainly used in two functions: to take note of data in the beginning of the prescription, and thus to prepare their use; and in the formulation of the division problem, "what may I posit to *b* which gives me *a*", with the answer "posit *p*; raise *p* to *b*, *a* it gives to you" (with minor variations). In the latter case, insertion into a computational scheme is likely to be meant.[39] Some texts also "posit" an "equalside" and its "counterpart", i.e., two sides of a square; the same process is indicated in other texts by the verbs *lapātum*, "to inscribe", or *nadûm*, "to lay down". In the geometric text BM 15285 there is no doubt that the actual sense of the latter term is to lay down *in drawing*; most likely, its general meaning in mathematical texts is "to lay down in writing or drawing". *lapātum* is also used regularly about numbers that afterwards serve in additive and subtractive operations; since these seem to have been performed on a counting board and not in clay [Høyrup 2000b], it may also have referred to recording on such a device, which is still in agreement with the general meaning of the term, "to grasp/take hold of"; "inscription" of coefficients, on the other hand, most likely refers to their being written down on a tablet for rough work, cf. [Robson 1999: 30].

At specific phrase for recording an (invariably intermediate) result is *rēška*

[39] In YBC 6504, g̃ar is used to take note of both intermediate and final results.

likīl, "may your head hold (it)". It seems to be reserved for numbers that are not to be inserted in a fixed scheme and therefore are not "posited".

The appearance of a result may be announced in several ways. It may be said that a number *illiakkum*, "comes up for you", from *elûm*, or that a calculation "gives" a certain result (*nadānum*, sum); alternatively, the text may state that "you see" the result (*tammar*, from *amārum*, "to see"). Very often, the calculation is simply followed by the enclitic particle *-ma* and the number, or by nothing but the naked number.

Third-millennium terminology

We have few mathematical texts from the third millennium, and our direct evidence for the corresponding technical terminology is correspondingly weak. We know that the verb si_8 was used at least since c. 2600 BCE to express that a segment λ was the side of the corresponding square area; uš, "length", sağ, "width" and $aša_5$ (=GÁN = IKU), "surface", can be followed back to 2400 BCE;[40] the use of the phrase igi *n* ğál is also documented for *n* = 3, 4 and 6 since c. 2400 BCE (cf. above).

The only mathematical documents from the Ur III period that contain terms for mathematical *operations* are the tables of reciprocals and of multiplication, of which the former use igi *n* ğál and the latter a.rá, "steps of".[41] The stable and invariably Sumerian terminology of the tables of square and cube roots, $íb.si_8$ and $ba.si_8$, allows us to conclude that these terms, too, will have been used already in Ur III.[42]

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[40] Texts in [Allotte de la Fuÿe 1915: 124–132]. For non-rectangular fields, these surveying texts distinguish uš and uš 2.kam, "2nd length", and sağ an.na and sağ ki.ta, "upper" and "lower width". The equality of (e.g.) lengths is expressed uš si_8. a.šà is used about the area in Sargonic texts [Whiting 1984: 69].

[41] Until very recently it was impossible to establish with certainty that any of the extant specimens were really of Ur III date; I have now been told by Eleanor Robson (personal communication) that at least tables of reciprocals have been found in dated UR-III contexts.

[42] The aberrant use of $ba.si_8$ in Eshnunna and Ur, it is true, could suggest that the distinction which all other text groups uphold between $íb.si_8$ and $ba.si_8$ is a secondary development, and perhaps (namely if we believe that early Old Babylonian Ur is a better witness of Ur-III usages than Eshnunna, only submitted to Ur III until

The other mathematical documents from the epoch, accounts and model documents, only give results, and tell neither the details of calculations nor the terminology in which these were spoken about.

A hymn in the praise of King Šulgi relates that the scribe school is a place where zi.zi.i g̃á.g̃á are learnt together with šid, "counting", and níg.šid, "accounting".[43] The use of the reduplicated forms g̃á.g̃á and zi.zi.i may perhaps depend on the context (description of a habitual practice and not of the single operation). g̃á.g̃á is the *marû* (approximately = imperfective/durative) stem of g̃ar, "to place" [SLa, 305], later used logographically for *šakānum*, "to posit"; in good agreement with the meaning of *kamārum* for which g̃ar.g̃ar is used logographically in the Old Babylonian age (namely "to place in layers, to accumulate"), g̃á.g̃á may thus be understood as "ongoing placing" – but also as "habitual placing". zi.zi is the *marû* stem of zi, "to rise, to stand up" [SLa, 322], and may perhaps be understood as "take up from" – not too far removed from *nasāḫum*, "to tear out", for which zi is used logographically in Old Babylonian texts,[44] nor however too close. g̃ar or g̃á.g̃á and zi or zi.zi may therefore have been the standard terms for addition and subtraction in the Ur III school; it is not to be excluded, however, that the reference is more specifically to "placing" on the counting board and "taking up" from it.

Other terms are not mentioned in the text, not even a term for multiplication, even though multiplication was certainly a cornerstone in the accounting system. However, the relation between syllabic and logographic writings of technical terms in Old Babylonian mathematical texts may permit us to approach the question from the opposite side.

First there are the terms that are written invariably (or almost so) with Sumerograms: uš, sag̃ (including sag̃ an.na and sag̃ ki.ta) and a.šà, when the "lengths", "widths" and "surfaces" of quadrangular and triangular

2025) that the form originally connected with the function as a verb was ba.si$_8$. In Eshnunna, it might then have displaced íb.si$_8$ even when used as a noun; in other groups, the term of the tables might have got the upper hand.

[43] "Šulgi-Hymn B", l.17, ed. Castellino 1972: 32].

[44] zi.zi is used in the unpublished text IM 121613 (courtesy of Jöran Friberg and Farouk al-Rawi) but does not appear in any published Old Babylonian text.

fields are meant;[45] and a.rá. To these come those which occur alternatingly as Sumerograms and as Akkadian loanwords (which indicates that they were really spoken with the Sumerian phonetic value): igi/*igûm* (with the cognates igi.bi/*igibûm* and igi.gub/*igigubbûm*); and íb.si$_8$/ib.sí/ba.si$_8$/ ba.se.e/.../*basûm* when not used logographically for the square configuration. Both of these categories, though used in the algebraic texts, have their roots in a much simpler and much more utilitarian calculational practice.

Then there are terms that may appear in syllabic as well as in logographic writing: *waṣābum*/daḫ, "to append"; *kamārum*/ĝar.ĝar/UL.GAR, "to accumulate", with *kumurrûm*/*kimrātum*/ĝar.ĝar/UL.GAR, "accumulation"; *nasāḫum*/zi, "to tear out"; *eli* ... *watārum*/ugu ... dirig, "over ... go beyond"; *našûm*/íl/nim, "to raise"; *esēpum*/tab, "to double"; *šutakūlum*/ì.gu$_7$.gu$_7$/ du$_7$.du$_7$/UR.UR/NIGIN, "to make hold each other"; *šutamḫurum*/NIGIN, "to make confront itself", with *mitḫartum*/LAGAB/NIGIN/íb.si$_8$; *paṭārum*/du$_8$, "to detach" (an igi); *šakānum*/ĝar, "to posit", with *nadānum*/sum, "to give", in the answer to the "division question"; *mišlum*/šu.ri.a, "a half" (together with other terms for simple fractions); *ḫepûm*/gaz, "to break", together with *bāmtum*, the "natural half", for which the logogram for ½ or šu.ri.a appear in a small number of texts that insist on writing as much as at all possible with logograms.

Finally there is a cluster of terms that occur only in Akkadian, or where logograms are either late or arguably not Sumerograms proper but artificial constructions which may have been recognized as such at the times.

Only one term for an operation belongs to this cluster, namely *ḫaraṣum*, "to cut off". It is more common than *nasāḫum*, "to tear out", in the early Akkadian groups ("group 1" and the Eshnunna group) but appears to have fallen disuse afterwards, perhaps because it occupied the same conceptual niche as *nasāḫum* which was well provided with a standard logogram (zi). In contrast, most terms belonging to the metalanguage are of this kind (those designating variables not, as discussed above).

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[45] In contrast, *šiddum* may replace uš when the length of a wall or a carrying distance is meant; similarly, saĝ may occur as rēšum, "head", when an initial value is intended, and is mostly substituted by the latter when an intermediate result is to be kept in memory. This excludes that the logogram is used simply because it is more easily written.

According to Old Babylonian lexical lists, *šumma*, "if", could be replaced by u_4.da ("at the day when") or tukun.bi, of which at least the former is no more difficult to write than the syllabic writing; but in all its mathematical functions, the word always remains syllabic.

inūma, "as", might also have been replaced by u_4.da; *aššum* might have been mu, as it actually happens in the Late Babylonian but pre-Seleucid mathematical text W 23291; but in Old Babylonian mathematical texts neither is ever written logographically.

inanna, "now", used to divide general from specific information in the statement, could have been i.ne.šè – but it always remains syllabic. So do the terms that serve to demarcate divisions within a prescription (*saḫārum*, "to turn around", and *târum*, "to turn back"), except in the late texts from Sippar. Lexical lists give niĝin as well as niĝin for either.

-ma, sometimes used to form "equations" where the right-hand side is a number, has no proper Sumerian equivalent, and often appears within text that are otherwise written in more or less grammatical Sumerian. *kīma*, "as much as", the indication that two non-numerical quantities are equally great, is replaced in the late series texts but nowhere else by gin_7.(nam), the Sumerian equative suffix. Also in the series texts but nowhere else, the "algebraic bracket" *mala*, "so much as", appears as a.na.

Among the interrogatives, *mīnûm*, "what", is written logographically as en.nam in the early Ur texts UET V, 121, 859 and 864; in several later texts (groups 3 and 6, and some scattered texts) the same equivalence is used; it is absent from Eshnunna. For the accusative *mīnâm*, UET 859 uses a.na.àm once, and so does IM 55357 from Eshnunna (asking for "what is equal", that is, for the side of a square). a.na.àm is also given by lexical lists, and is regular Sumerian meaning "what is it" ([SLa, §120] quotes it from *Inanna's Descent*); there is no need to postulate any direct connection between the two appearances. en.nam, as far as I have been able to find out, has no antecedents before the Old Babylonian epoch; moreover, [AHw, 656a] only records it as used in mathematical texts. All in all, it seems to be an *ad hoc* construction made in the context of early Old Babylonian mathematics teaching, probably in Ur[46]. *kī maṣi* has no logographic equivalent in the

[46] Since it occurs together with a.na.àm in UET V, 859 but in a distinct function, it is unlikely to be a mere phonetic writing of that term.

mathematical texts, even though the similar phrase *mala maṣi* used in legal texts [AHw, 621b][47] looks like an Akkadian translation of a.na.àm. *kiyā*, the question for several values, is similarly deprived of logographic equivalent (but seems to be so in general, not only within the mathematical corpus).

Since the semantic boundaries between *šakānum*, *lapātum* and *nadûm* is difficult to establish, the apparent absence of logographic writings for the last two terms may not be significant – we cannot exclude that g̃ar is used more broadly than *šakānum* (though no positive evidence suggests it to be). More interesting are the terms that announce results.

Of these, *nadānum*, "to give", was already mentioned for having the logogram sum in connection with the "division question". In this function, the term is used in all text groups, either in logographic or in syllabic writing; as a general marker of results it is only found in group 3 and (as *nadānum*) in AO 6770 (group 1) and in the two texts YBC 4662 and 4663, in which some slips indicate that it has been introduced as a substitute for a phrase *tammar* used in an original from which they were copied with terminological revisions. In my [2000a: 165] I summed up the conclusions that can be drawn from this contribution as follows:

> The origin of nadānum as a marker of results is with multiplications in the sexagesimal system (and thus with the post-Šulgi tradition); the main use of the term in texts where it does not serve for results in general is in the division question; second comes the indication of the result of "raising" multiplications; thirdly that of final results. It is probably significant that even group 3 tends to use the syllabic forms in this original domain.
>
> It is also significant that the use of *nadānum* goes together with the only systematic breaking of the normal switch between grammatical persons.[48] This switch between the statement formulated by an "I" and a prescription of what "you" should do (at times involving quotes from what "he" has said) becomes a general characteristic of mature Old Babylonian mathematics, the only texts that have some difficulty with the principle being indeed those of group 1. But if it is systematically broken in connection with a term pointing towards the post-Šulgi school tradition, it must come from elsewhere – and the format of [many of the

[47] It is also found (in the subjunctive form *mala maṣû* in the mathematical text VAt 7531.

[48] [Namely the question "what may *I* posit to *b* which gives *me a*", with answer "posit *p*; raise *p* to *b*, *a* it gives to *you*"].

Eshnunna] texts shows how it fits the riddle tradition: The "he" of the prescription is indeed the "someone" who has posed the problem and the "I" of the statement, whence different from the speaking person of the prescription who explains the procedure.

elûm, "to come up", has no Sumerographic equivalent at all when used about results (when used instead of *našûm*, "to raise", it is usually written nim.[49] This is quite striking: in Eshnunna, where it is used alternatingly with *tammar*, it is indeed predominantly linked with such computations as point toward the Ur-III scribal tradition. In view of the invariably Akkadian writing we can be fairly certain that the idiom of results "coming up" is a post-Ur III contribution to the scribal computational tradition – probably made in the periphery or the northern part of the core (Nippur) since the term is absent from all texts from the core area with the exception of three group-1 texts and the Nippur texts.

tammar, "you see", *may* on rare occasions appear in logographic writings: pàd ("to see") in early Ur, igi.dù in IM 55357, igi.du_8 ("to look at") in the series texts YBC 4669 and 4673, igi in VAT 672. Apart from the appearance in Ur and the apparent slips in YBC 4662 it is totally absent from the core area whereas it dominates the periphery (in Eshnunna used alongside with "coming up", "seeing" belonging predominantly in problems of riddle character, "coming up" as stated above rather with the traditional utilitarian scribal type). The absence from the core is noteworthy, since the parlance can be seen to have been known even there: Str 366 and VAT 7531, both from group 3 (the group in which the "giving" of results was adopted generally) refer to it obliquely in phrases "in order that I see [as result]", how much "do I see".

The fluctuation of the logographic equivalents for *tammar* is best explained if we suppose the Akkadian phrase to be primary and the logograms to be translations from this. One thing must be observed, however: results are already "seen" in a group of Sargonic school texts about rectangles and squares – see [Whiting 1984: 59 n.2]). In these, the word is pàd. This is no proof (and hardly circumstantial evidence) that the use of pàd is continuous:

[49] The only Old Babylonian text which has a Sumerographic writing of "coming up" is the Nippur text CBM 12648, which appears to construct Sumerograms for everything, and which has íb.si_8 x e_{11}.dè, "make the equalside of x come up".

pàd simply happens to be the most precise Sumerian translation of the Akkadian verb *amārum*;[50] moreover, *if* pàd had been carried by the tradition, there would have been no reason to invent a Sumerian substitute in Eshnunna. But it is fairly hard evidence that the Akkadian and the Old Babylonian parlance is linked in one of the two languages, and thus probably through an Akkadian (whence necessarily non-scribal) tradition – which fits the apparent bond between "seeing" and the riddle tradition perfectly well.

Educating the Kapo

I shall not repeat the arguments I have presented elsewhere in favour of the thesis that the particular pattern of Old Babylonian mathematics arose from a synthesis between a tradition of scribal computation deriving from the Ur-III (and, only rarely to be distinguished, pre-Ur-III[51]) scribal computation and one or more traditions of non-scribal practitioners' mathematics – the latter contributing with mathematical riddles that might serve the display of professional proficiency, in agreement with the ideals of Old Babylonian scribal "humanism" (nam-lú.ulù). The above presents only a small part of the evidence; for further elaboration of the argument I shall only refer to my [2000a; forthcoming/a; forthcoming/b]. Instead I shall concentrate on what vindicates Robert Englund's unpleasant view of Ur III – a view so unpleasant that many Assyriologists prefer not to believe in its veracity.

Let us first look at what happened outside Ur when the pattern of Old Babylonian mathematics was formed. Apart from some modest vacillation in Eshnunna it appears to have been accepted by everybody that the standard mensurational terms (uš, sag, including "upper" and "lower", and a.šà) and the core vocabulary for using the place value system (a.rá,

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[50] By the same argument, the unorthographic igi.dù, the corresponding orthographic igi.du_8, and the ellipsis igi – less adequate and all presumably from the northern periphery – are more likely to be connected.

[51] As I have argued elsewhere, the use of íb/ba.si_8 as a verb is likely to derive from pre-Ur-III usages, while the treatment of the term as a noun reflects its appearance as an "object" in the tables that became increasingly important after the Ur-III introduction of the place value system.

igi, igi.gub) referred to legitimately Sumerian practices and should hence be expressed in Sumerian; even in Eshnunna, most deviations from the norms prevailing elsewhere take the shape of unorthographic Sumerian or loanwords, syllabic Akkadian is much rarer.

As far as mathematical operations are concerned, it depends on the general style of the text whether they are spoken of in syllabic Akkadian or logograms. In a number of cases, logograms appear to be fresh inventions, either based on puns (e.g., ì.gu$_7$.gu$_7$), on geometric (e.g., LAGAB) or semantic-grammatical similarity (reduplication of the sign rendering reciprocity); in others, traditional Sumerian names for operations were employed, even though their non-technical meaning did not coincide precisely with that of the Akkadian terms for which they served as logograms, at times in spite of disagreement with the preferences of lexical lists (zi for *nasāḫum*, g̃ar.g̃ar for *kamārum*). In some cases, logograms were chosen for which we have no evidence (nor, however, counter-evidence) that they had been used in earlier times as mathematical terms, but where the choice is semantically meaningful (daḫ for *waṣābum*, tab for *esēpum*, gaz for *ḫepûm*[52]). Some schools may have had syllabic writing as their stylistic preference and others may have favoured logographic writing. Their way to speak of mathematical operations was then governed by this general canon of style; the texts betray no tendency overriding general preferences to favour or to avoid referring to the operations in Sumerian.

When turning to the metalanguage we encounter a wholly different situation. None of the "logical operators" *šumma*, *inūma* and *aššum* are ever written logographically; of the terms that allow to formulate a specific question on the background of general information or to structure a complex procedure (*inanna*, *saḫārum* and *târum*), only the last is replaced by a logogram, and this only happens in the late group 6. Similarly, those which allow the formulation of an equation (*-ma*, *kīma* and *mala*) always remain syllabic outside the late group of series texts, written in very compact pseudo-Sumerian. Among the interrogatives, *kī/mala maṣi* and *kiyā* always remain syllabic, and *mīnûm* always so except for one appearance of a.na.àm in Eshnunna and from its replacement in the late texts from group 6 by what

[52] But lexical lists also give the equivalences gaz-*nasāḫum*, tab-*waṣābum*, which is no less meaningful.

may at the time have been recognized as a specifically mathematical pseudo-Sumerogram. Finally, among the terms that announce a result, only the one connected to place-value multiplications (that is, to a practice with a clear Sumerian history) is ever written logographically, except for the utterly scarce appearances of igi.du_8 for *tammar*, all but one of which are moreover late.

Since logograms for most of these terms were available in lexical lists (whence familiar to the authors and users of the mathematical texts), we must conclude that they were consciously evaded. But these are the terms that are needed if mathematics teaching is to go beyond the inculcation of routines, that is, if it is to be based on problems. We must conclude that the authors of the mathematical texts chose to demonstrate through the way they formulated problems that *problems were a new genre* and no inheritance from the Sumerian scribe school. And problems, as is well known, constituted the core and most of the body of Old Babylonian mathematics.

So far I have left out early Old Babylonian Ur from this analysis. Ur, the former queen of Ur III, of course was not likely the emphasize the break with the past, even though the sparse problem texts that have been found do suggest that they come from a borrowed tradition.[53]

If is therefore not astounding that the metamathematical vocabulary of these texts is in Sumerian, but noteworthy that it is meagre and not very consistent: en.nam (probably a new pseudo-Sumerogram, cf. above) is used in UET V, 121, 864 (in varying spellings) and 859, and a.na.àm once in UET V, 859. Results are "seen" (pàd) in UET V, 859 and 864, whereas they carry the Sumerian enclitic copula -àm,⟨ "it is",[54] in UET V, 858. The hyperorthographic (whence not traditional – íb.tag_4 does not occur in the

[53] In UET V, 864, everything is in Sumerian except the key technical terms dakāšum and dikištum; the topic was thus something which could not be discussed in Sumerian, obviously because it had not been done before; UET V, 858 presents us with the problem of a bisected trapezium, but in a trite version that suggests that the author had heard that the transversal 13 of a trapezium with parallel sides 17 and 7 divides the height in ratio 2:3 but either did not know *that* or did not know *why* this transversal bisects the area. That it did was known in Sargonic surveying and constituted stock knowledge in Old Babylonian mathematics in general; it can only have been found out by somebody who also knew the an argument *why*.

[54] Never used elsewhere in this function in Old Babylonian mathematics!

perfectly parallel form ì.ib.tag₄) prospective prefix ù.ub., "and then", appears to be a fresh translation of the Akkadian suffix *-ma*. This certainly does not look like a continuation of a stable tradition, rather as a hesitating exploration of an unknown terrain.

Before drawing the final conclusions we might ask whether Old Babylonian mathematics teachers are really likely to have expressed any ideas or ideals through their shaping and use of a technical terminology. The answer is emphatically affirmative – examples of very conscious adoption of a characteristic terminological canon and of elimination of unwanted terms abound. One example was mentioned above: the elimination of the familiar phrase *tammar* from group 3 and YBC 46627, 4663 and its replacement by *nadānum*/sum (elsewhere exclusively used in connection with sexagesimal multiplication). We may also mention AO 8862 from group 1, which asks questions about the dimensions of fields by *mīnûm*, whereas *kī maṣi* is used when brick calculations are concerned. Most striking of all is perhaps a group of 9 tablets found in the same room in Tell Harmal, Eshnunna, which invariably couple results "coming up" with the question *mīnûm* and the "seeing" of results with *kī maṣi* (a system which is found nowhere else and even broken in a tablet coming from the neighbouring room – see [Høyrup 2000a]: 122 n.9, 126).

Deliberate shaping of the vocabulary beyond what was required by the mathematics it was meant to express was thus not alien to the mind-set of Old Babylonian mathematics teachers, and we may trust that they had an idea when requiring one category of terms to be written invariably with Sumerian logograms, allowing others to appear in syllabic or logographic writing according to the local stylistic canon, and still others to be written always in syllabic Akkadian. Seeing that those which are invariably in Sumerian are legitimately so from the historical point of view, we may conclude that those which are not allowed to appear in Sumerian – those that allow the formulation and solution of *problems* – were supposed to represent something new; the meta-terminological vacillations of the problem texts from Ur indicate that the mathematics teachers of the remaining Old Babylonian orbit were not mistaken on this account. The whole *discourse of problems* must have been absent from the legacy left by the Ur III school.

Mathematical problems, of course, were not a fresh invention of the Old Babylonian school. Students of the Old Akkadian schools had learned their mathematics from problems, and "saw" results; problems are also to be found in the school texts from Shuruppak. What will have been new is instead a mathematics teaching *not* based on problem solution, a mathematics teaching deliberately eliminating even the slightest appeal to independent thought on the part of the students. It is also unique in history: even the school of the Third Reich went no further in its control of the students' minds than to having problems deal with "artillery trajectories, fighter-to-bomber ratios and budget deficits accruing from the democratic pampering of hereditarily diseased families" [Grunberger 1974: 367].

In that school which came out of an administrative reform which completed a military reform, Ur III seems to have aimed at exactly what Orwell's [1954: 241] *Newspeak* was meant to effectuate: "to make all [unauthorized] modes of thought impossible" – at least among those miserable and terrible Kapo–overseer-scribes who constituted the "Outer Party" of king Šulgi's statal system.

List of tablets referred to

AO 6770. Published in [MKT II 37f, cf. III 62*ff*].

AO 8862. Published in [MKT I 108–113, II Taf. 35–38; III 53].

BM 13901. Published in [Thureau-Dangin 1936].

BM 15285. Published in [MKT I 137]; with an additional fragment in [Saggs 1960]; with yet another in [Robson 1999: 208–217].

BM 85200+VAT 6599. Published in [MKT I 193*ff*, II Pl 7–8 (photo), Pl 39–40 (hand copy)].

BM 96957+VAT 6598. Published in [Robson 1999: 231–244].

CBM 12648. Published in [MKT I 234*f*], much improved in [Friberg 2001: 149].

CBS 154+921. Published in [Robson 2000: 40].

CBS 19761. Published in [Robson 2000: 36*f*].

IM 52301. Published in [Baqir 1950b].

IM 52916+52685+52304. Published in [Goetze 1951].

IM 55357. Published in [Baqir 1950a].

IM 121613. Preliminary publication in [Friberg & al-Rawi 1994].

Str 366. Published in [MKT I 257, III 56], hand copy [Frank 1928: #9 = pl vii].

Str 367. Published in [MKT I 259*f*], hand copy [Frank 1928: #10 = pl viii].

Str 368. Published in [MKT I 311], hand copy [Frank 1928: #11 = pl ix].

TMS V. Published in [TMS 35–49, Pl 7–10].

TMS VI. Published in [TMS 49–51, Pl 11–13].

TMS XVI. Published in [TMS 91*f*, Pl 25].

TMS XXVI. Published in [TMS 124*f*, Pl 39].

UET V, 121. Published in [Vajman 1961: 248], cf. [Friberg 2000: 35*f*].

UET V, 858. Published in [Vajman 1961: 251], cf. [Friberg 2000: 38*f*].

UET V, 859. Published in [Vajman 1961: 254*f*], cf. [Muroi 1998: 201*f*] and [Friberg 2000: 39].

UET V, 864. Published in [Vajman 1961: 257*f*], cf. [Muroi 1998: 200] and [Friberg 2000: 37]

VAT 672. Published in [MKT I 267, II Taf 43].

VAT 7531. Published in [MKT I 289*f*, II Taf 46; III 58].

VAT 7532. Published in [MKT I 294*f*, II Taf 46; III 58].

VAT 7535. Published in [MKT I 303–305, II Taf 47].

W 23291. Published in [Friberg 1997].

YBC 4662. Published in [MCT 71*f*, Pl 8].

YBC 4663. Published in [MCT 69, Pl 7].

YBC 4669. Published in [MKT I 514, III 27*f*, Taf 3].

YBC 4673. Published in [MKT I 507*f*, II 508, III 29–31, Taf 3].

YBC 6504. Published in [MKT III 22*f*, Taf 6].

YBC 7997. Published in [MCT 98, Pl 23].

YBC 9856. Published in [MCT 99, Pl 4].

YBC 9874. Published in [MCT 90, Pl 11].

References

AHw: Wolfram von Soden, *Akkadisches Handwörterbuch*. Wiesbaden: Otto Harrassowitz, 1965–1981.

Allotte de la Fuÿe, François-Maurice, 1915. "Mesures agraires et formules d'arpentage à l'époque présargonique". *Revue d'Assyriologie* 12, 117–146.

Alster, Bendt, 1974. *The Instructions of Šuruppak. A Sumerian Proverb Collection.* (Mesopotamia: Copenhagen Studies in Assyriology, 2). Copenhagen: Akademisk Forlag, 1974.

Baqir, Taha, 1950a. "An Important Mathematical Problem Text from Tell Harmal". *Sumer* 6, 39–54.

Baqir, Taha, 1950b. "Another Important Mathematical Text from Tell Harmal". *Sumer* 6, 130–148.

Bruins, Evert M., 1971. "Computation in the Old Babylonian Period". *Janus* 58, 222–267.

Diakonoff, I. M. (ed.), 1969. *Ancient Mesopotamia, Socio-Economic History. A Collection of Studies by Soviet Scholars*. Moskva: »Nauka« Publishing House, Central Department of Oriental Literature.

Englund, Robert K., 1990. *Organisation und Verwaltung der Ur III-Fischerei*. (Berliner Beiträge zum Vorderen Orient, 10). Berlin: Dietrich Reimer.

Frank, Carl, 1928. *Straßburger Keilschrifttexte in sumerischer und babylonischer Sprache*. (Schriften der Straßburger Wissenschaftlichen Gesellschaft in Heidelberg, Neue Folge, Heft 9). Berlin & Leipzig: Walter de Gruyter.

Friberg, Jöran, & Farouk al-Rawi, 1994. "Equations for Rectangles and Methods of False Value in Old Babylonian Geometric-Algebraic Problem texts". *Manuscript in progress*, courtesy of the authors.

Friberg, Jöran, 1997. "'Seed and Reeds Continued'. Another Metro-Mathematical Topic Text from Late Babylonian Uruk". *Baghdader Mitteilungen* 28, 251–365, pl. 45–46.

Friberg, Jöran, 2000. "Mathematics at Ur in the Old Babylonian Period". *Department of Mathematics, Chalmers University of Technology, The University of Göteborg. Preprint* 2000/25.

Friberg, Jöran, 2001. "Bricks and Mud in Metro-Mathematical Cuneiform Texts", pp. 61–154 *in* Jens Høyrup & Peter Damerow (eds), *Changing Views on Ancient Near Eastern Mathematics*. (Berliner Beiträge zum Vorderen Orient, 19). Berlin: Dietrich Reimer.

Goetze, Albrecht, 1951. "A Mathematical Compendium from Tell Harmal". *Sumer* 7, 126–155.

Grunberger, Richard, 1974. *A Social History of the Third Reich*. Harmondsworth, Middlesex: Penguin.

Hallo, William W., 1976. "Toward a History of Sumerian Literature", pp. 181–203 *in Sumerological Studies* in Honor of Thorkild Jacobsen on his Seventieth Birthday, June 7, 1974, (The Oriental Institute of the University of Chicago, Assyriological Studies, 20). Chicago & London: University of Chicago Press.

Høyrup, Jens, 1980. "Influences of Institutionalized Mathematics Teaching on the Development and Organization of Mathematical Thought in the Pre-Modern Period. Investigations into an Aspect of the Anthropology of Mathematics". *Materialien und Studien. Institut für Didaktik der Mathematik der Universität Bielefeld* 20, 7–137.

Høyrup, Jens, 1990a. "Algebra and Naive Geometry. An Investigation of Some Basic Aspects of Old Babylonian Mathematical Thought". *Altorientalische Forschungen* 17, 27–69, 262–354.

Høyrup, Jens, 1990b. "Sub-Scientific Mathematics. Observations on a Pre-Modern Phenomenon". *History of Science* 28, 63–86.

Høyrup, Jens, 1992. "The Babylonian Cellar Text BM 85200 + VAT 6599. Retranslation and Analysis", pp. 315–358 *in* Demidov et al (eds), *Amphora. Festschrift für Hans Wussing zu seinem 65. Geburtstag*. Basel etc.: Birkhäuser.

Høyrup, Jens, 2000a. "The Finer Structure of the Old Babylonian Mathematical Corpus. Elements of Classification, with some Results", pp. 117–177 *in* Joachim Marzahn & Hans Neumann (eds), *Assyriologica et Semitica*. Festschrift für Joachim Oelsner anläßlich seines 65. Geburtstages am 18. Februar 1997. (Altes Orient und Altes Testament, 252). Münster: Ugarit Verlag.

Høyrup, Jens, 2000b. "A Note on Old Babylonian Computational Techniques". *Filosofi og Videnskabsteori på Roskilde Universitetscenter*. 3. Række: *Preprints og Reprints* 2000 Nr. 3.

Høyrup, Jens, forthcoming/a. *Lengths – Widths – Surfaces. An examination of Old Babylonian algebra and its kin*. To be published by Springer Verlag (2001?).

Høyrup, Jens, forthcoming/b. "On a Collection of Geometrical Riddles and Their Role in the Shaping of Four to Six 'Algebras'". To appear in *Science in Context*, 2001.

MCT: O. Neugebauer & A. Sachs, *Mathematical Cuneiform Texts*. (American Oriental Series, vol. 29). New Haven, Connecticut: American Oriental Society, 1945.

MKT: O. Neugebauer, *Mathematische Keilschrift-Texte*. I-III. (Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik. Abteilung A: Quellen. 3. Band, erster-dritter Teil). Berlin: Julius Springer, 1935, 1935, 1937.

Muroi, Kazuo, 1998. "Early Old Babylonian Mathematical Problems Written in Sumerian". *Historia Scientiarum*, Second Series 7:3, 199–203.

Nissen, Hans J., Peter Damerow & Robert Englund, 1990. *Frühe Schrift und Techniken der Wirtschaftsverwaltung im alten Vorderen Orient. Informationsspeicherung und -verarbeitung vor 5000 Jahren*. Bad Salzdetfurth: Verlag Franzbecker.

Nissen, Hans J., Peter Damerow & Robert Englund, 1994. *Archaic Bookkeeping: Writing and Techniques of Economic Administration in the Ancient Near East*. Chicago: Chicago University Press.

Orwell, George, 1954. *Nineteen Eighty-Four*. Harmondsworth, Middlesex: Penguin.

Robson, Eleanor, 1999. *Mesopotamian Mathematics 2100–1600 BC. Technical Constants in Bureaucracy and Education*. (Oxford Editions of Cuneiform Texts, 14). Oxford: Clarendon Press.

Robson, Eleanor, 2000. "Mathematical Cuneiform Tablets in Philadelphia. Part 1: Problems and Calculations". *SCIAMUS* 1, 11–48.

Saggs, H. W. F. 1960. "A Babylonian Geometrical Text". *Revue d'Assyriologie* 54, 131–146.

Sjöberg, Åke W., 1976. "The Old Babylonian Eduba", pp. 159–179 *in Sumerological Studies* in Honor of Thorkild Jacobsen on his Seventieth Birthday, June 7, 1974, (The Oriental Institute of the University of Chicago, Assyriological Studies, 20). Chicago & London: University of Chicago Press.

SLa: Marie-Louise Thomsen, *The Sumerian Language. An Introduction to its History and Grammatical Structure.* (Mesopotamia, 10). København: Akademisk Forlag, 1984.

Steinkeller, Piotr, 1979. "Alleged GUR.DA = u g u l a - g é š - d a and the Reading of the Sumerian Numeral 60". *Zeitschrift für Assyriologie und Vorderasiatische Archäologie* 69, 176–187.

Thureau-Dangin, F., 1936. "L'Équation du deuxième degré dans la mathématique babylonienne d'après une tablette inédite du British Museum". *Revue d'Assyriologie* 33, 27–48.

TMS: E. M. Bruins & M. Rutten, *Textes mathématiques de Suse.* (Mémoires de la Mission Archéologique en Iran, XXXIV). Paris: Paul Geuthner, 1961.

Vajman, A. A., 1961. *Šumero-vavilonskaja matematika. III-I Tysjačeletija do n. e.* Moskva: Izdatel'stvo Vostočnoj Literatury.

Whiting, Robert M., 1984. "More Evidence for Sexagesimal Calculations in the Third Millennium B.C." *Zeitschrift für Assyriologie und Vorderasiatische Archäologie* **74**, 59–66.

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