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TERTIUM NON DATUR
or
On reasoning styles in early mathematics

Jens Høyrup

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The arguments of the following paper are largely distilled from a variety of topics I have worked on over the years; in the interest of relative brevity I have been forced to leave out almost all of the factual background for the conclusions I have drawn on earlier occasions. To an unpleasant extent, the bibliography is therefore dominated by my own publications; further references to sources and to works of other scholars are found in these.

Two convenient scapegoats

In [1972: 3, 14], Morris Kline wrote the following lines:

> Mathematics as an organized, independent, and reasoned discipline did not exist before the classical Greeks of the period from 600 to 300 B.C. entered upon the scene. There were, however, prior civilizations in which the beginnings or rudiments of mathematics were created.
>
> ...
>
> The question arises as to what extent the Babylonians employed mathematical proof. They did solve by correct systematic procedures rather complicated equations involving unknowns. However, they gave verbal instructions only on the steps to be made and offered no justification of the steps. Almost surely, the arithmetic and algebraic processes and the geometrical rules were the end result of physical evidence, trial and error, and insight.

Such blunt statements (as well as the less blunt but similar attitudes of many fellow writers) have called forth objections from other quarters.[1] An as example one may quote George Gheverghese Joseph's statement [1991: 89*f*] that if "the Greek dependence on Egypt and Babylonia is now recognized, the myth of the 'Greek miracle' will no longer be sustainable".[2] Unfortunately for Joseph's intended undermining of "one of the central planks of the Eurocentric view of history of progress" [1991: 90], the whole discussion of Egyptian and Babylonian mathematics is nothing but support for Kline's view.[3] Admittedly, Richard

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[1] In some sense these anti-Eurocentric objections have often been paradoxical, their aim being to show that "non-Western" cultures had the same kind of (meta-)mathematics as the Greeks: implicitly, the ideals of (what we find in) Greek mathematics are accepted.

As I shall argue in the end, certain mathematical cultures (not ethnic but professional cultures) have had the attitude that under particular circumstances some mathematics should *not* be reasoned, and have had it for a good reason.

[2] Elsewhere, indeed, Joseph [1991: 125–129] goes into direct though imprecise polemic with Kline.

[3] It is immaterial for the present purpose that they are often awfully wrong in details (terribly wrong datings, freely invented "translations", confusion of modern interpretation and ancient text, similar confusion between algorithm and theoretical algebra – see [Høyrup 1992]) and thus allow opponents of the author's general aim to conclude that no good arguments can be found in favour of the existence of non-Greek, non–Greek-

Gillings [1972: 233] is quoted to the effect that a

> nonsymbolic argument or proof *can* be quite rigorous when given for a particular value of the variable; the conditions for rigor are that the particular value should be *typical*, and that a further generalization to *any* value should be *immediate*

– but Joseph does not show that (nor discuss in which sense) the various *rules* applied to particular cases he quotes from Egyptian and Babylonian material can really be read as paradigmatic (or "potentially general") "argument or proof" in Gillings's sense.

In the following sections of the paper I shall show that much of Old Babylonian mathematics *was* indeed reasoned in this sense; characterize the type of reasoning involved; confront it with Euclidean reasoning about analogous cases; use this to characterize the approach of Greek theoretical geometry as embodied by Euclid's *Elements*; and briefly discuss a different type of Greek mathematical reasoning. In the end I shall widen the perspective toward other mathematical cultures.

Old Babylonian geometric proto-algebra

Kline as well as Joseph speak about "Babylonian mathematics" as if this entity remained the same as long as the Babylonian culture lasted; so did until very recently almost everybody who dealt with the topic without being a specialist of exactly this historical field. At closer inspection, however, there are important differences between the mathematics of the Old Babylonian and the Seleucid periods (c. 1900–1600 BCE and c. 300–100 BCE, respectively). The large majority of texts comes from the Old Babylonian period, on which I shall concentrate at first.

The Old Babylonian mathematical corpus consists of three parts: tables, tablets for rough numerical work, and problem texts. Only the last group is relevant for the present discussion – actually only the "procedure" texts which prescribe how to solve the problem stated in the beginning.

A large part of the problem texts have been understood since they were first interpreted in the 1930s to be of "algebraic" character.[4] Taken at their words, most of them deal with the measurable sides and areas of rectangles and squares, but these were taken to serve as mere dummies for unknown numbers and their products. Correspondingly, the operations that were performed were supposed to be arithmetical additions, subtractions, multiplications, etc. In this reading, the procedure descriptions look like mere prescriptions of numerical algorithms, with no indication of the way these have been found. A historian like Otto Neugebauer, who knew the corpus well, was fully aware that they could not have been found without genuine mathematical reasoning, and presupposed that the texts had gone together with a system of oral instruction explaining the reasons for the steps; those general historians who knew only one or two simple examples in translation often believed that they had been found by trial and error (Kline, as we see, combines the two ideas).

Only a thorough investigation of the structure of the terminology and of the discursive

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derived mathematics. In the view of anybody who shares the aim, this is of course the most serious shortcoming of the book.

[4] The history of these interpretations is described in [Høyrup 1996a].

organization of the texts reveals that the texts have to be taken at their geometrical words.[5] The problems are indeed (in a loose sense) homomorphic with those of numerical equation algebra, but many of the operations are geometric, not arithmetical.

As a first example we may look at the text YBC 6967[6], which contains a single problem dealing with two numbers *igûm* and *igibûm* belonging together in the table of reciprocals, "the reciprocal and its reciprocal". This problem thus illustrates another respect in which the technique is similar to modern equation algebra: a functionally abstract "basic representation" (with us abstract numbers, with the Babylonians measured or measurable segments and areas) is used to represent magnitudes belonging to other ontological domains but involved in relations that are structurally similar to those characterizing the basic representation.

The text goes as follows in literal translation:

Obv.

1. [The *igib*]*ûm* over the *igûm*, 7 it goes beyond
2. [*igûm*] and *igibûm* what?
3. Yo[u], 7 which the *igibûm*
4. over the *igûm* goes beyond

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[5] The first thorough exposition of this analysis is [Høyrup 1990]; equally thorough but probably more reader-friendly is [Høyrup 2002].

Part of the outcome of the structural analysis (and one of the reasons that the arithmetical interpretation breaks down) is the sharp distinction between two different additive operations (not merely synonyms for the same operation), between two different multiplicative operations, two different halves, and no less than four different "multiplications". Since we shall encounter the additions below, they may serve as example. One of them I shall translate "appending", the other "accumulation". The former stands for a concrete joining to a magnitude which conserves its identity (in the same sense as addition of the interest conserves the identity of my bank account – interest on a loan *is* indeed called "the appended" in Babylonian); the other may be used about the purely arithmetical addition of the measuring numbers of ontologically different magnitudes – e.g., of lengths and areas, of areas and volumes, or of men, days, and bricks carried by the men in question during the days in question.

[6] Based on the transliteration in [Neugebauer & Sachs 1945: 129]; as everywhere where no translator is indicated the English translation is mine. The numbers are expressed in a sexagesimal place value system (that is, a system with base 60), in which ´, ˝, ... indicate decreasing and `, ``, ... increasing sexagesimal order of magnitude (and ° when needed "order zero"); 30´ is thus $30{\cdot}60^{-1} = ½$, 15´ = $15{\cdot}60^{-1} = ¼$. These indications of absolute order of magnitude are not present in the original – the number notation of the mathematical texts (obviously not that of accounting and practical surveying!) is a floating-point system.

Words in [ ] are damaged on the tablet and reconstructed from parallel passages; words in ( ) are added for comprehensibility.

5. to two break:[7] 3°30´;
6. 3°30´ together with 3°30´
7. make hold:[8] 12°15´.
8. To 12°15´ which comes up for you
9. [1` the surf]ace append: 1`12°15´.
10. [The equalside[9] of 1`]12°15´ what? 8°30´.
11. [8°30´ and] 8°30´, its counterpart,[10] lay down.

Rev.

1. 3°30´, the made-hold,
2. from one tear out,
3. to one append.
4. The first is 12, the second is 5.
5. 12 is the *igibûm*, 5 is the *igûm*.

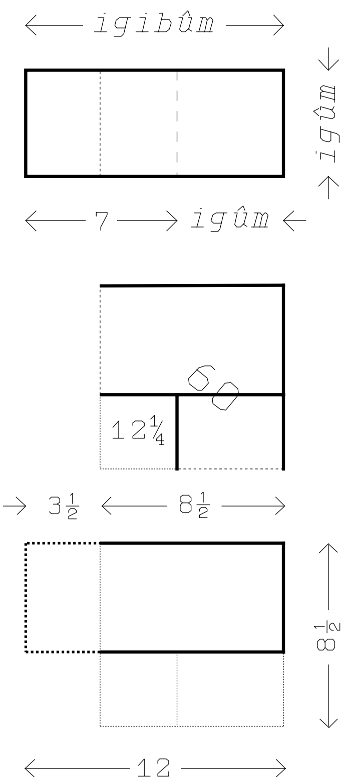

Figure 1. The representation of the *igûm-igibûm* problem of YBC 6967.

What goes on may be followed in the diagram of Figure 1. We should expect the product of the two numbers to be 1, but it is actually meant to be 60 (whether due to the floating-point character of the number system or to the origin of the table of reciprocals as a tabulation of aliquot parts of 60 is uncertain). The two numbers are thus represented by the sides of a rectangle with area 1` (as obvious, e.g., from the reference to 1` in obv. 9 as a "surface". Since we are told that the *igibûm* exceeds the *igûm* by 7, the length of the rectangle exceeds the width by 7. This excess (with appurtenant section of the rectangle) is bisected, and the outer part moved around so as to contain together with the inner part a square □(3½), whose area will be 12¼. When the original rectangle (transformed into a gnomon) is joined to this, a square with area 60+12¼ = 72¼ is produced. The "equalside" of this area is 8½, and so is its "counterpart". When that part of the rectangle which was "made hold" is restored to its original position, we get the original length, the *igibûm*, which will thus be 8½+3½ = 12. But before

[7] "Breaking" is a bisection that produces a "necessary half", a half that could not have been chosen differently – e.g., that half of the base of a triangle that serves in area calculation. On the other hand, if a problem states that a square area and a half of the side are accumulated, the other, "accidental" half occurs – it might just as well have been a third.

[8] "Making *a* and *b* hold" stands for the construction of the rectangle contained by the sides *a* and *b* – henceforth ⊏⊐(*a*,*b*).

[9] The "equalside of *A*" (in the terms of other texts, that which "is equal along *A*") is the side of *A* when this area is laid out as a square; numerically it corressponds to the square root of *A*.

[10] The "counterpart" of an "equalside" is the side with which it has a corner in common.

we can restore it, we must remove it from the place where it was put; this removal produces the *igûm*, which must therefore be 8½–3½ = 5.

As we see, no attempt is made to discuss *why* or *under which conditions* the operations performed are legitimate and lead to the correct result. On the other hand it is intuitively obvious, once we are familiar with the properties of rectangles, that everything *is* correct. In this sense the prescription is, as formulated by Karine Chemla ([1991, 1996], and elsewhere) regarding Chinese mathematics, algorithm and proof in one.

The clay tablet contains no drawing; a few others do, but only as support for the statement, never as a supplement to the prescription. For this reason we cannot know the precise character of the diagrams that supported the reasoning – they may have been drawn in sand strewn on a brick floor, on a wall, or in any other medium that has not been conserved; we do not even known to which extent trained calculators would make actual drawings, and to which extent they would rely on mental geometry. We may be confident, however, that drawings were made use of at some stage of the instruction – mental geometry builds on previous experience with material geometry, just as mental addition of multi-digit numbers presupposes previous exposure to pen-and-paper algorithms for almost all of us; we may also be fairly confident that the diagrams in case were structure diagrams and not made carefully to scale – field plans, at least, had this character (see Figure 2, a plan from the 21st century BCE). As we see, only the right angles (those angles which are essential for the determination of areas) are rendered correctly; in general, the Babylonians seem not to have regarded angles as quantifiable magnitudes – expressed in a pun, an angle which was not "right" was simply considered "wrong".

The notion of a "naive" proof integrated in the algorithm may astonish us, but should not do so. How, indeed, will we normally treat the corresponding problem in symbolic algebra if we merely need to solve it? More or less in the following steps:

$$x-y = 7 \qquad xy = 60 \tag{1}$$

$$\frac{x-y}{2} = 3\tfrac{1}{2} \tag{2}$$

$$\left(\frac{x-y}{2}\right)^2 = 12\tfrac{1}{4} \tag{3}$$

$$\left(\frac{x-y}{2}\right)^2 + xy = 12\tfrac{1}{4}+60 = 72\tfrac{1}{4} \tag{4}$$

$$\left(\frac{x+y}{2}\right)^2 = 72\tfrac{1}{4} \tag{5}$$

$$\frac{x+y}{2} = \sqrt{72\tfrac{1}{4}} = 8\tfrac{1}{2} \tag{6}$$

$$x = \frac{x+y}{2}+\frac{x-y}{2} = 8\tfrac{1}{2}+3\tfrac{1}{2} = 12 \tag{7}$$

$$y = \frac{x+y}{2}-\frac{x-y}{2} = 8\tfrac{1}{2}-3\tfrac{1}{2} = 5 \tag{8}$$

We would obviously be able to justify every step if asked by somebody who did not follow the idea – but we would hardly justify the step from (3) to (4) with exact reference to the appropriate Euclidean axiom (or corresponding arithmetical theorem or axiom). Just as the Babylonian calculator, we thus proceed

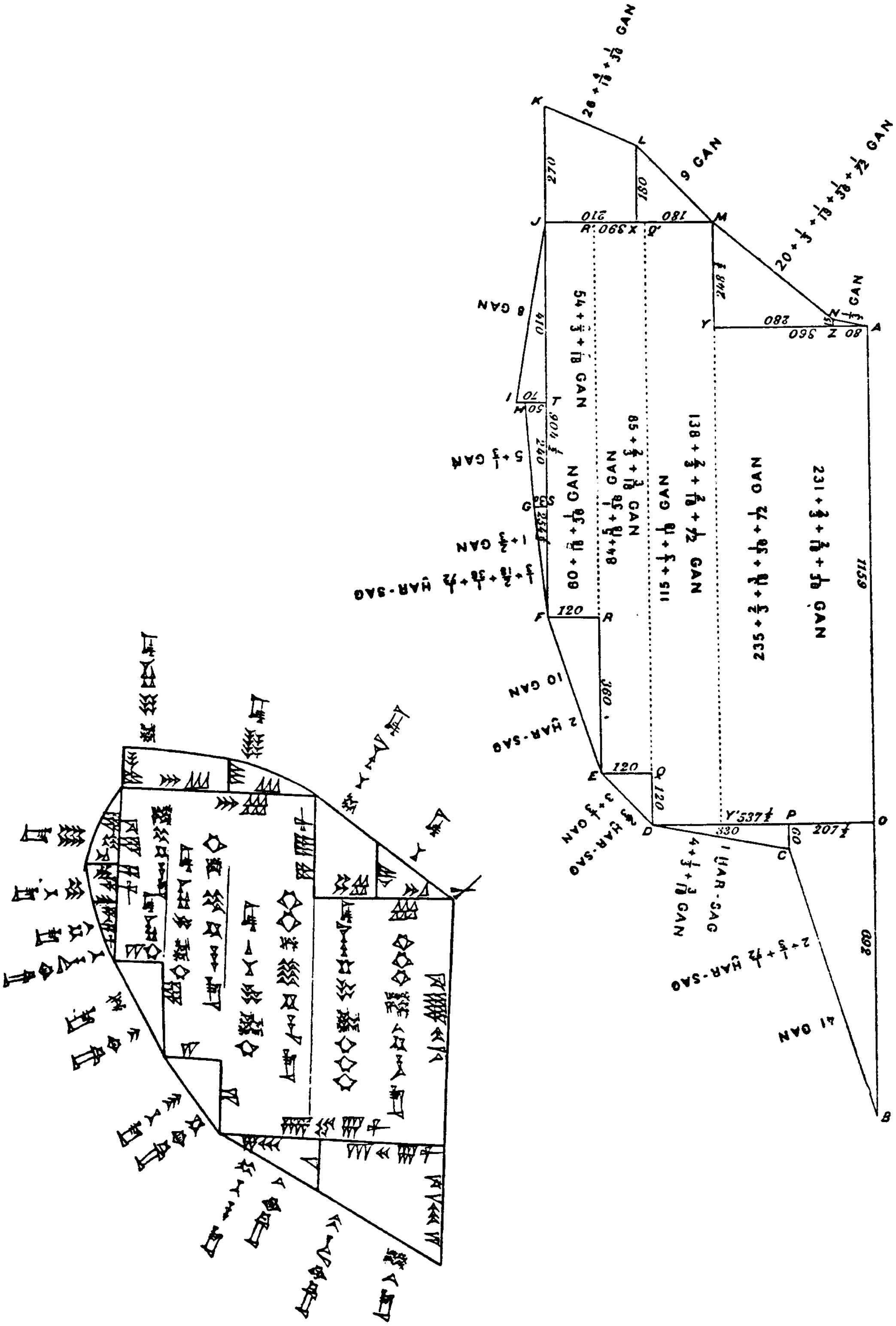

Figure 2. Field plan as drawn on the tablet (left) and in true proportions (right). From [Thureau-Dangin 1897: 13, 15].

naively; so did any equation algebra until the advent of the Modern era. And just as that of the Babylonian calculator, our approach is *analytic:* we take the existence of the solution for granted, manipulate it as if it were known, and stop when we have disentangled the unknowns from the complex relationships in which they were involved.

Whereas the geometrical diagrams on which the reasoning was made have not survived, a few texts have transmitted the kind of explanations which will normally have been given orally. All are from Susa, a peripheral area (which may be the reason that explanations which elsewhere were transmitted within a stable oral tradition had to be put into writing). One – TMS XVI – explains the transformations of two linear equations.[11] The first transformation runs as follows in translation:[12]

1. [The 4th of the width, from] the length and the width to tear out, 45´. You, 45´
2. [to 4 raise[13], 3 you] see. 3, what is that? 4 and 1 posit,[14]
3. [50´ and] 5´, to tear out, ⌈posit⌉. 5´ to 4 raise, 1 width. 20´ to 4 raise,
4. 1°20´ you ⟨see⟩, 4 widths. 30´ to 4 raise, 2 you ⟨see⟩, 4 lengths. 20´, 1 width, to tear out,
5. from 1°20´, 4 widths, tear out, 1 you see. 2, the lengths, and 1, 3 widths, accumulate, 3 you see.

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[11] The use of the term "equation" is no anachronism. The equations of a modern engineer or economist state that the measure of some composite magnitude equals a certain number, or that the measure of one magnitude equals that of another; exactly the same is done in the Babylonian texts.

[12] Based on the hand copy and transliteration in [Bruins & Rutten 1961: 91*f*, pl. 25], with corrections from [von Soden 1964]. Cf. revised edition of the full tablet in [Høyrup 1990: 299–302]. The translation in the original edition should be used with caution, and the commentary is best disregarded completely.

[13] "Raising" designates the determination of a concrete magnitude by means of a multiplication, and presupposes a consideration of proportionality. Originally the metaphor referred to the determination of a prismatic volume with height *h*, obtained by "raising" the base from its virtual height of 1 cubit (presupposed by the metrology, which measured volumes in area units) to the real height.

[14] "Positing" appears to mean "taking note of" materially, at times on a counting board, at times by writing a length along a line as in Figure 2.

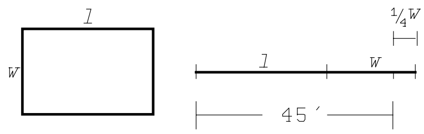

Figure 3. The situation of TMS XVI #1.

6. Igi 4 de[ta]ch,[15] 15´ you see. 15´ to 2, lengths, raise, [3]0´ you ⟨see⟩, 30´ the length.

7. 15´ to 1 raise, [1]5´ the contribution of the width. 30´ and 15´ hold.

8. Since "The 4th of the width, to tear out", it is said to you, from 4, 1 tear out, 3 you see.

9. Igi 4 de⟨tach⟩, 15´ you see, 15´ to 3 raise, 45´ you ⟨see⟩, 45´ as much as (there is) of [widths].

10. 1 as much as (there is) of lengths posit. 20, the true width take, 20 to 1´ raise, 20´ you see.[16]

11. 20´ to 45´ raise, 15´ you see. 15´ from $^{30}{}_{15}$´[17] [tear out],

12. 30´ you see, 30´ the length.

The equation deals with the length (ℓ) and the width (w) of a rectangle – see Figure 3; in the actual case, however, this concrete meaning is relatively unimportant. In line 1, we are indeed told (in symbolic translation) that

$$(\ell+w)-\tfrac{1}{4}w = 45' .$$

At first we are instructed to multiply the right-hand side by 4, from which 3 results. In line 2, the meaning of this number is asked for; the explanation given in lines 2–5 can be confronted with Figure 4, which may correspond more or less closely to something the author had in mind, and which is anyhow useful for us. As we observe, no problem is solved, the explanations presuppose (and the student is thus supposed to know) that the length is 30´ and the width 20´, their sum 50´ and the fourth of the width 15´.

In line 6, the equation is multiplied by ¼, from which follows both the "contribution of the width", that is, the value of the member (1–¼)w, and the *coefficients* ("as much as there is") of length and width.

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[15] igi n designates the reciprocal of n. For numbers where this was possible, division by n was performed as a raising to igi n (in administrative calculation it was always possible, since all technically relevant coefficients were rounded to numbers that possessed a convenient igi).

Finding igi n was spoken of as "detaching" it; the idea was probably that one part was detached from a bundle of n parts of unity.

[16] This step may refer to a distinction between a "real" field with dimensions 30 and 20 (180 m × 120 m, since the tacitly presupposed length unit was the "rod" equal to c. 6 m) and a "model field" 30´ × 20´, i.e., 3 m × 2 m, certainly more easily drawn in the school yard; since the text does not indicate absolute order of magnitude this must remain a hypothesis.

[17] This renders the non-standard way () in which "45" is written in this place in the tablet.

Two other didactical expositions are found in the text TMS IX #1 and #2.[18] Both deal with geometry of the kind that was used to represent the *igûm* and *igibûm* in YBC 6967. They run as follows:

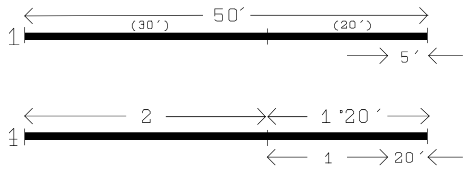

Figure 4. The transformations of TMS XVI #1.

#1

1. The surface and 1 length accumulated,
4[0´. ¿30, the length,? 20´ the width.]
2. As 1 length to 10´ ⌈the surface, has been appended,]
3. or 1 (as) base to 20´, [the width, has been appended,]
4. or 1°20´ [¿is posited?] to the width which 40´ together ⌈with the length ¿holds?]
5. or 1°20´ toge⟨ther⟩ with 30´ the length hol[ds], 40´ (is) [its] name.
6. Since so, to 20´ the width, which is said to you,
7. 1 is appended: 1°20´ you see. Out from here
8. you ask. 40´ the surface, 1°20´ the width, the length what?
9. [30´ the length. T]hus the procedure.

#2

10. [Surface, length, and width accu]mulated, 1. By the Akkadian (method).
11. [1 to the length append.] 1 to the width append. Since 1 to the length is appended,
12. [1 to the width is app]ended, 1 and 1 make hold, 1 you see.
13. [1 to the accumulation of length,] width and surface append, 2 you see.
14. [To 20´ the width, 1 appe]nd, 1°20´. To 30´ the length, 1 append, 1°30´.
15. [¿Since? a surf]ace, that of 1°20´ the width, that of 1°30´ the length,
16. [¿the length together with? the wi]dth, are made hold, what is its name?
17. 2 the surface.
18. Thus the Akkadian (method).

In #1, as we see, we are told that the arithmetical sum of the length and the area of a rectangle is $A+\ell$ = 40´; once again, the explanation of what goes on presupposes the student to know that the length is 30´ and the width 20´. The text then explains how this is to be given a concretely meaningful interpretation. The trick is to replace the length ℓ by a rectangle ⊏⊐(1,ℓ), which corresponds to joining an extra "base 1" to the width, as shown in Figure 5 (the orientation of which follows from the designation of the extension as a "base"). The resulting total "width" is 1°20´; since the total area is 40´, this is seen to correspond to the length 30´, as it should.

In #2, we are told instead the arithmetical sum of the length, the width and the area, $A+\ell+w$ = 1. Once again, the dimensions are presupposed to be known, ℓ = 30´, w = 20´, as can be seen in line 14. This time we are told to add ⊏⊐(1,1) = 1 to the sum $A+\ell+w$; the result is then shown to be the area of a new rectangle with length

[18] Based on the transliteration and hand copy in [Bruins & Rutten 1961: 63*f*, pl. 17], with corrections from [von Soden 1964]. Cf. revised edition in [Høyrup 1990: 320–323]. Even in this case, the translation and the commentary in the original edition ask for benign neglect.

L = 1+30´ = 1°30´, width W = 1+20´ = 1°20´ – cf. Figure 6.[19] This section of the text is said to explain the "Akkadian method"; since the trick that distinguishes #2 from #1 is the joining of a quadratic complement to a (pseudo-)gnomon, the "Akkadian method" is likely to be exactly this trick, basic for the solution of all mixed second-degree problems.

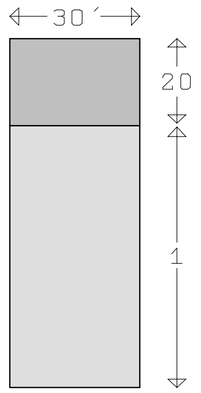

Figure 5. The configuration described in TMS IX #1.

#3 of the tablet, the last problem and a problem in the proper sense, combines the equation of #2, $A+\ell+w = 1$, with an equation of the same type as the one explained in TMS XVI though more abstruse – namely

$$\frac{1}{17}(3\ell+4w)+w = 30' \ .$$

This is reduced, this time without didactical explanation, to

$$3\ell+21w = 8°30' \ ,$$

after which the corresponding equation for "the length and width of the surface 2" (L and W) is derived,

$$3L+21W = 32°30' \ .$$

Since $\sqsubset\!\sqsupset(L,W) = 2$, $\sqsubset\!\sqsupset(3L,21W)$ is found to be $2{\cdot}3{\cdot}21$ = 1`3 (i.e., 63), and in the end the resulting rectangle problem for $\Lambda = 3L$, $\Omega = 21W$,

$$\Lambda+\Omega = 32°30' \ , \quad \sqsubset\!\sqsupset(\Lambda,\Omega) = 1`3$$

(the additive analogue of the problem solved in YBC 6967) is solved, and first L and W, next ℓ and w are found. No didactical explanation of how to solve the rectangle problem is extant, but we may safely assume that such an explanation was at hand and that its style was similar to what we know from TMS XVI and TMS IX #1–2.

Before we leave the Old Babylonian period it should be pointed out that certain aspects of the procedure descriptions reflect the presence of "critique", that question for the reasons for and the limits of the validity of the procedure which is the antithesis of the "naive" approach. One is the precedence of "tearing-out" over "appending" in YBC 6967, rev. 2–3, the other the explicit introduction of the "base 1" in TMS IX #1.

That these features of the text are "critical" only becomes visible when the historical development of Old Babylonian "algebra" is understood, which requires another structural analysis of the corpus, this time associating the distribution of

[19] This presence of several "lengths" and "widths" shows why the exposition needs to presuppose that the measures of the configuration are known: these measuring numbers serve as identifying tags, and are needed for this purpose in the absence of letter or similar symbols.

Even many genuine problem texts refer to the value of certain entities before they are found. This may give the impression that the problems are overdetermined and their authors hence mathematically incompetent. This, however, is a mistaken reading: the information which is made use of never exceeds what is necessary; this constitutes the set of "given numbers", which is always kept strictly apart from those numbers which are "merely known" and used as identifiers.

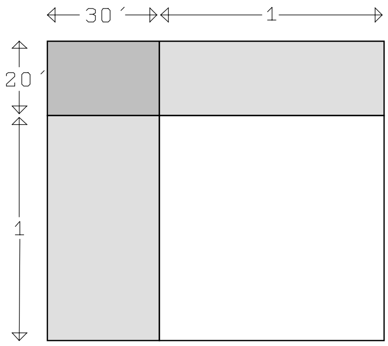

Figure 6. The configuration of TMS IX #2.

synonyms and characteristic phrases with orthography and what (little) is known about the archaeological provenience of tablets (most, indeed, have been bought by museums on the antiquity market), and correlation of the problems found in the Old Babylonian corpus with those found in a number of other historical contexts (Seleucid and other Late Babylonian problem texts, ancient Greek theoretical, Neopythagorean and practitioners' mathematics, Arabic algebra and agrimensorial texts, Jaina and Italian *abbaco* mathematics). I shall not attempt to reduce the necessary complex arguments to what can be contained in a few paragraphs[20] but only sum up the relevant results.

In the later third and incipient second millennium BCE, a restricted number of geometrical riddles circulated in a lay (that is, non-scribal, non-schooled) and probably Akkadian-speaking[21] environment of surveyors / practical geometers. A number of these were to be solved by means of the kind of naive cut-and-paste geometry which we have encountered in YBC 6967 and by application of the trick of the quadratic completion (thus for good reason designated the "Akkadian method"; the trick seems to have been discovered at some moment before c. 1900 BCE, and probably after c. 2200 BCE): to find the side of a square from the sum of the side or "all four sides" and the area, or from the difference one or the other way around; to find the sides of a rectangle from the area and the diagonal or from the area together with the sum of or difference between the sides (with a few variants); problems dealing with two concentric squares (with given sum of / difference between the sides and the areas) were apparently solved by means of standard diagrams.

In the nineteenth century BCE, these problems were adopted into the Old Babylonian scribe school, where they gave rise to the development of the so-called "algebra" (which is much more refined than can be seen from the above examples, solving mixed third-degree problems by means of factorization – reducing biquadratic problems and even a bi-biquadratic problem stepwise; inverting the role of unknowns and coefficients; etc.). As it turns out, those text groups which are closest to the lay tradition do not respect the "norm of concreteness" according to

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20 The structural analysis of the corpus is described in [Høyrup 2000b], and (with some extensions and minor revisions) in [Høyrup 2002: 317–361]. The place of Old Babylonian "algebra" in the network of mathematical cultures was first investigated in [Høyrup 1996b]; a more thorough exposition is [Høyrup 2001]. Information on the latter topic is also given in [Høyrup 2002: 362–417, *passim*].

21 The hegemonic and scribe school language of the third millennium was Sumerian. However, the presence of Akkadian (later split into a Babylonian and an Assyrian dialect) is attested already before 2500 BCE, gradually rising to become the dominant language in the early second millennium. With extremely few exceptions the language of the Old Babylonian mathematical texts is Akkadian, though the writing often makes heavy use of Sumerian word signs (as English writing may make use of the medieval word sign for Latin *videlicet*, rendered as *viz* yet presupposing a pronunciation "namely").

which "tearing-out" must precede "appending" of the same entity but use the elliptic phrase "append and tear out"; some early texts, moreover, follow the habit of many lay surveying traditions from elsewhere and operate with a notion of "broad lines", that is, with the idea that a line carries an inherent standard width[22]. For this reason, they are able to "append" sides to areas, which indeed they do.

The school environment, however, appears to have found it difficult to accept the conflation of linear and planar extension, and therefore formulated the inhomogeneous sums as "accumulations" (namely, of the measuring numbers), devising moreover a variety of designations for the standard width which transforms a side into a rectangular area[23]. Some schools also seem to have found it absurd to "append" something which is not yet at hand, and therefore introduced the "norm of concreteness". If "critique" is understood as investigations of *why* and *under which conditions* our usual maive ways and conventional wisdom hold good,[24] then these are full-blown examples.

The chronological dissection of the Old Babylonian corpus allows a final observation of importance for our topic.[25] All above examples were formulated around paradigmatic cases, though in agreement with Gillings's criteria for when an argument from a paradigmatic case can be considered rigorous – cf. p. 2. This is no accident: almost all Old Babylonian mathematical texts that present us with explicit or implicit arguments have this character. There are, however, exceptions, and a few texts do indeed formulate rules in general terms. These rules may build on insight and argument, and can hardly have been invented without the intervention of some kind of mathematical insight; the rules themselves, however, only prescribe steps to be performed, and contain no trace of an argument.[26] Interestingly, all such attempts at general formulation belong in the earliest texts. The way such rules turn up in later sources suggest that they were a borrowing from the lay tradition, within which they may indeed have been very useful.[27] Within the school, however, they were soon eliminated, being both ambiguous when not supported by an example

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22 As does cloth today, when we buy "three yards of curtain material". The notion of the "broad line" and its appearance in a number of practical geometries is examined in [Høyrup 1995].

23 One of these designations is the "base" of TMS IX #1; but at least two alternatives are attested in the corpus.

24 "Untersuchung der Möglichkeit und Grenzen derselben", as expressed in Kant's *Critik der Urtheilskraft* (B III [Werke V, 237]).

25 See [Høyrup 2002: 344, 383, and *passim*].

26 Nor should they, this is not the nature or purpose of a rule – our multiplication table contains no hint of the role of associativity and distributivity of the operations involved.

27 The general rule is an adequate tool for an oral tradition, being more easily remembered mechanically and transmitted faithfully than the full paradigmatic example; explanations and examples can then be improvised once the master knows what is meant by a possibly ambiguous rule. A parallel is offered by the relation between fixed formulae and relatively free use of these by the singer in oral epic poetry, see [Lord 1960: 99–102 and *passim*]).

and pedagogically useless (probably because they were deprived of argument). The absence of abstract general rules is thus, like the compliance with the norm of concreteness, no consequence of a primitive mind unable to free itself from concrete thought; to the contrary, both have resulted from deliberate pedagogical or philosophical choice.

Euclidean geometry

Figure 1 is quite similar to the diagram of *Elements* II.6 – see Figure 7. Since the underlying mathematical structures are also analogous (to the extent a problem can be analogous with a justification of the way it is solved), it seems obvious to look closer at this Euclidean proposition.

In the Thomas Heath's faithful translation [1926: I, 385] it states the following:

> If a straight line be bisected and a straight line be added to it in a straight line, the rectangle contained by the whole with the added straight line and the added straight line together with the square on the half is equal to the square on the straight line made up of the half and the added straight line.

Next follows what Antiquity would apparently see as a particular example with indubitable paradigmatic value[28] but which Kline (and most modern readers) have come to regard as actually and not only potentially general:[29]

> For let ⟨any⟩ straight line *AB* be bisected at the point *C*, and let ⟨any⟩ straight line *BD* be added to it in a straight line; I say that the rectangle contained by *AD*, *DB* together with the square on *CB* is equal to the square on *CD*.

The proof starts by constructing the latter square (*CEFD*) and drawing the diagonal *DE*. Next through *B* the line *BHG* is drawn parallel to *CE* or *DF* (*H* being the point where the line cuts *DE*) and through *H* the line *KM* parallel to *AB* or *EF*. Finally, through *A* the line *AK* is drawn parallel to *CE* or *DF*.

Now the diagram is ready, and with reference to the way the construction was made ⊏⊐*AL* is shown to equal ⊏⊐*HF*. Adding ⊏⊐*CM* to both, the gnomon *CDFGHL* is seen to equal ⊏⊐*AM*. Further addition of ☐*LG* shows that ⊏⊐*AM* together with ☐*LG* equals ☐*CF*, as stated in the theorem.

The second part of the proof follows the pattern of the cut-and-paste procedure of YBC 6967 precisely. The important difference is the presence of the first part. Thanks to this, things are not just "seen", they are as firmly established as required by the norms of Greek geometry – we do not move areas around and glue them together, we *prove* that one area (⊏⊐*AL*) is equal

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[28] Apart from the use of the habitual format rule–example and the precise wording, this interpretation is supported, for instance, by Aristotle's analogous reference to geometric arguing which is correct if only we avoid including in the premises we draw on the particular characteristics of the drawing made on the ground (*Metaphysics* M, 1078ᵃ19–20). See also the detailed discussions in [Mueller 1982: 11–14] and [Netz 1999: 247-258]

[29] Trans. [Heath 1926: I, 385], with minor corrections in ⟨⟩.

to another (⊏⊐*HF*). Even the fact that the gnomon *CDFGHL* together with □*LG* is identical with □CF), though not argued in detail, could be proved rigorously by repeated use of proposition II.1.

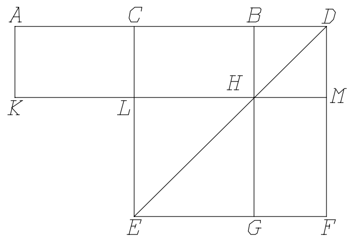

Figure 7. The diagram of *Elements* II.6.

The first part of the proof of proposition II.6 can thus be seen as a *critique* which consolidates the well-known. Other propositions and proofs from the sequence *Elements* II.1–10 invite to make similar observations and interpretations. To this we may add that the riddles of the surveyors' tradition were doubtlessly known in classical Antiquity – as we shall see (below, p. 19), the riddle of "the four sides and the area" turns up in the pseudo-Heronian *Geometrica*. The whole sequence repeats matters that were familiar in the surveyors' tradition at least since the earliest second millennium BCE; many of the propositions, moreover, are never used explicitly later on in the work, which supports the interpretation that their critical consolidation was an aim in itself. Finally, all are proved independently, although a derivation of one from the other would often have been easy (actually, II.5 and II.6 are equivalent, and so are II.9 and II.10); what needs to be consolidated is thus not only the customary knowledge contained in the propositions but also the traditional naive-geometric *argument*.[30]

Greek theoretical geometry as a whole was evidently much more than a consolidation of the well-known; in as far as its ideals of what constitutes a *proof* are concerned, however, book II of the *Elements* may be regarded as representative. In aiming at critique of the already familiar it is certainly no first in the history of mathematics – as we have seen, something similar was made in the Old Babylonian scribe school, and it is part of the dynamics of any institutionalized teaching of mathematics at levels where appeals to the reasoning of the students are required.[31] In the Old Babylonian school, however, the role of critique had been peripheral and accidental; in Greek

[30] Being necessarily ignorant of the whole prehistory, Heath [1926: I, 377] formulated this as follows:

> What then was Euclid's intention, first in inserting some propositions not immediately required, and secondly in making the proofs of the first ten practically independent of each other? Surely the object was to show the power of the *method* of geometrical algebra as much as to arrive at results.

[31] This topic is dealt with in [Høyrup 1985], and, more crudely but more precisely in aim and with broader historical scope, in [Høyrup 1980].

theoretical it was, if not *the* very centre then at least an essential gauge.[32]

Stations on the road

In the Old Babylonian mathematical texts we find names for particular lines, but we find no term for a linear extension in general; nor is any term for an angle (or right angle) to be found. This does not mean that surveyors could not speak about lines unless they were already defined as the length or width of a field, the length or height of a wall, a carrying distance, etc., nor that they were unable to refer to the corner of a building; but *tubqum* ("corner") was not used as a technical term in mathematics. In general, it is doubtful whether the terminology of Old Babylonian mathematics can at all be characterized as "technical". Instead, as concluded in [Høyrup 2002: 302],

> it is rather a very standardized use of everyday language to describe an extra-linguistic – computational and naive-geometrical – practice which was always *more* standardized than the linguistic description. The linguistic description was thereby analogous to our heuristic explanations in standardized ordinary language of what goes on in those symbolic formulae which with us constitute the level of real technical operation.

An early step in the unfolding of Greek theoretical critique was the establishment of definitions. Irrespective of Aristotle's claim that Socrates "was the first to concentrate upon definition",[33] discussions of semantic delimitations go back as far in Greek (proto-)philosophy as we can follow it – a very early example is Hesiod's pointing out in *Works and Days* [ed., trans. Mazon 1979: 86] that the word "strife" (ἔρις) corresponds to two very different things (namely peaceful competition and cruel war). The definition of number as a "multitude composed of units"[34] is likely to go back at

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32 In the introduction to the *Method,* Archimedes argues that "we should give no small part of the credit to Democritus who was the first to make the assertion [that the cone is the third part of the cylinder, and the pyramid of the prism] though he did not prove it" [trans. Heath 1912: 13]. The rhetoric of the argument implies that the opposite attitude prevailed; rhetoric may distort things but becomes ineffective if the recipient knows that it is fully off the point – which we may therefore suppose that it was *not*, the recipient (Eratosthenes) being as conversant as anyone with both the mathematics and the norms that governed it at his times.

33 *Metaphysics* A, 987^{b}3, trans. [Tredennick 1933: I, 43]. The Greek term is ὁρισμός, related to the Euclidean term ὅρος, the former meaning something like "delimitation" / "marking out by boundaries", the latter "limit" / boundary.

34 Itself an outcome of critique, which remained fateful for more than 2000 years and encumbered the theoretical justification of *algebra* in the early Modern era, since this attempt to make unambiguous and stable sense of the notion of a number excluded both *1* (a fact which Euclid forgets when defining a "part" in *Elements* VII, immediately after he has repeated the habitual definition of a number!) but also fractions.

least to the fifth century BCE, and many other definitions were known to, and discussed by, Plato and Aristotle. Of particular interest are the definitions of the various classes of (rectilinear) angles [trans. Heath 1926: I, 181]:

10. When a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is *right* [...].
11. An *obtuse angle* is an angle greater than a right angle.
12. An *acute angle* is an angle less than a right angle.

These were known to Aristotle, who refers to them in *Metaphysics* M 1084^{b}7. But they may have been a relatively fresh invention in his days, since Plato's Socrates speaks in *Republic* VI, 510C [trans. Shorey 1930: II, 111] of the three kinds of angles as things of which geometers "do not deign to render any further account to themselves or others, taking it for granted that they are obvious to everybody".[35]

A clear notion of a right angle is evidently essential for making proofs like that of *Elements* II.6. In Aristotle's times the above definition was apparently supposed to be sufficient. This follows from can be derived from Aristotle's writings about the status of the Euclidean postulates. On the whole, he does not seem to have heard of them [McKirahan 1992: 133–137], which would suggest that their need had not yet been felt. Only the second postulate appears to have been known to Aristotle in a formulation close to what we find in the *Elements* – *Physics* III, 207^{b}29–31 [trans. Hardie & Gaye 1930] explains that mathematicians "do not need the infinite and do not use it. They postulate only that the finite straight line may be produced as far as they wish".

This implies that no need had as yet been discovered around the mid-fourth century for postulate 4, "that all right angles are equal to one another", and thus, since this principle is essential for a large number of proofs of the equality of figures, that it was tacitly believed to be inherent in the definition. In Euclid's time, on the other hand, it was recognized that this was not the case. Although Greek geometers of the early fourth century may have felt critique just as compulsory as their third-century successors, the level at which critique was actually performed was raised in the historical process – which of course cannot astonish if we recognize that mathematical rigour is a human product and never absolute.

[35] The passage may also mean, however, that they allow no further discussion *beyond* the definitions they have given, in which case the definitions will have been older.

Other Greeks

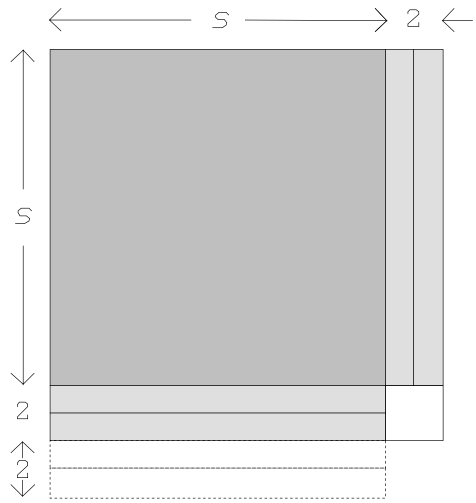

Figure 8. The procedure described in *Geometrica* 24.3.

The community of "theoreticians" (however that was delimited) was not the only community of the classical world to deal with mathematics. On one hand, the social need for mathematical practitioners was certainly not lower than it had been in the older Egyptian and Mesopotamian civilizations (nor probably significantly higher); on the other, the diffuse area encompassing Neopythagoreanism, Hermeticism, Gnosticism and Neoplatonism was also fond of mathematical metaphors and astounding mathematical insights.[36] In sources stemming from either community, instances or traces of mathematical reasoning can be located. In both cases what we find is naive, not critical. I shall present one example from each.

The first, belonging to the practitioners' tradition, comes from the pseudo-Heronian *Geometrica*.[37] It is a Greek version of the riddle of "all fours sides and the area":

> A square surface having the area together with the perimeter of 896 feet. To get separated the area and the perimeter. I do like this: In general (καθολικῶς, i.e., independently of the parameter 896 – JH), place outside (εκτίθημι) the 4 units, whose half becomes 2 feet. Putting this on top of itself becomes 4. Putting together just this with the 896 becomes 900, whose squaring side becomes 30 feet. I have taken away underneath (ὑφαιρέω) the half, 2 feet are left. The remainder becomes 28 feet. So the area is 784 feet, and let the perimeter be 112 feet. Putting together just all this becomes 896 feet. Let the area with the

36 [Cuomo 2000] is a pioneering investigation of the situation and interplay of these groups in late Antiquity, in particular as reflected in Pappos's *Collection*.

37 *Geometrica* 24:3, ed. [Heiberg 1912: 418], photographic reproduction of the manuscript [Bruins 1964: I, 53]. As with the Babylonian texts, my translation is meant to be pedantically literal. Actually, we should speak of "Heiberg's" rather than of any pseudo-Hero's *Geometrica*. Heiberg produced the bulk of the conglomerate from two ancient treatises which were already composite and cannot be traced back to a common source (as told quite explicitly by Heiberg, but in Latin and in a different volume of the Heronian *Opera omnia* [Heiberg 1914: xxiii–xxiv], for which reasons the fact has generally gone unnoticed). These two treatises are represented, respectively, by Heiberg's mss A+C and mss S+V. Chapters 22 and 24, however, are independent treatises (24 another conglomerate) which happen to be contained in the same codex as *Geometrica*/S but at a distance. See [Høyrup 1997: 77].

perimeter be that much, 896 feet.[38]

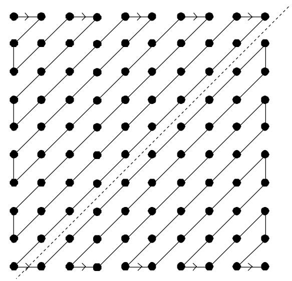

Figure 9. 10×10 arranged as a "race-course".

The procedure that is described is shown in Figure 8 (the manuscript only contains a drawing of a square with inscribed value for the side and the area; apparently, the geometry is meant to be either mental or performed independently by the reader[39]). As we see, the procedure is identical with what we have seen in the text YBC 6967, apart from those details that follow from the fact that we are dealing with a square and not with a rectangle. The style is certainly reasoned: "I have taken away underneath the half, 2 feet are left. The remainder [when these too are removed] becomes 28 feet"; but it is fully naive. The text also points out which numbers belong to the type *in general* (square area and perimeter) and do not depend on the particular parameters of the example as a way to safeguard potential generality; this is currently done in the various *Geometrica*-components and also in kindred medieval treatises, and already in one text from Old Babylonian Susa.

The various Neopythagorean writings are less generous when it comes to reveal the reasoning behind the mathematical facts they relate – maybe because astounding mathematical facts, once we understand their grounds, tend to be less astounding and therefore less serviceable for the display of wisdom beyond ordinary human reason. Sometimes, however, reasons shine through. One interesting case is found in Iamblichos's commentary to Nicomachos's *Introduction*[40]: namely the observation that 10×10 laid out as a square and counted "in horse-race" (see Figure 9) reveals that

$$10\times10 = (1+2+...+9)+10+(9+...+2+1)$$

whence

$$10\times10+10 = 2T_{10} ,$$

T_n being the triangular number of order n. This argument will have been common Pythagorean or Neopythagorean lore, if we

[38] Heiberg does not grasp the geometrical procedure that is described, for which reason his commentaries are misguided, imputing the faulty understanding on the ancient copyist.

[39] This is also the case in the *Liber mensurationum*, an Arabic treatise building on the surveyors' tradition (known from Gherardo da Cremona's Latin translation, ed. [Busard 1968]): the sequence of problems about squares starts by a drawn square, that of rectangle problems with a rectangle, etc. Only a few fourteenth- and fifteenth-century Latin and Italian descendants of the tradition contain drawings illustrating the whole procedure.

[40] Ed. [Pistelli 1975: 75$^{25–27}$], cf. [Heath 1921: 113*f*].

are to believe Iamblichos's exposition, though hardly a discovery made within this environ-
ment.[41] In any case, the naive type of reasoning will not have been left behind when the Pythagorean scientologists took over from existing mathematics that which they managed to understand (which could be neither the theory of *Elements* X, Apollonian *Conics*, Archimedean infinitesimal methods, nor "Heron's" formula for the triangular area).

Proportionality – reasoning and its elimination

Does this mean that mathematics is always in some way reasoned, either naively or critically? In some sense *yes*, simply because we are unlikely to count as "mathematics" activities which are wholly devoid of understanding, however much they have to do with countable items or take place in geometrical space. But mathematics need not always *be taught*, nor to *be exercised* as a reasoned practice. When learning to drive a car you probably received a number of instructions and explanations, about changing gears, about braking and aquaplaning, etc. But woe to your passengers if you use your conscious mental reserves too intensively on thinking about these matters when you move in the traffic.

A mathematician behaves no different. Most of the transformations of symbolic expressions are performed automatically, leaving energy for conscious reflection on the more intricate and still unfamiliar aspects of the problem that is treated; the activity of the mathematician thus remains reasoned, if only at a higher level.

But the routine activity of the mathematical practitioner may be different in character. Remaining in the pre-Modern epoch, we may illustrate this through a look at the way simple linear problems were dealt with.

A typical late medieval rule for solving such problems can be found in Jacopo da Firenze's *Tractatus algorismi* from 1307.[42] It runs as follows:

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[41] Firstly, in belongs squarely within the style of *psēphos* arithmetic that can be presupposed to be at the basis of the "doctrine of odd and even"; this was generally familiar at too early a moment to be Pythagorean – Epicharmos Fragment B 2 ([Diels 1951: I, 196; earlier than c. 475 BCE]) refers to the representation of an odd number ("or, for that matter, an even number") by a collection of *psēphoi* as something trivially familiar. Secondly, the ensuing formula for the triangular number,

$$T\mathrm{n} = \frac{n^2+n}{2} ,$$

belongs no less squarely within a cluster of summation formulae shared between Seleucid and Egyptian Demotic sources which betray no Greek influence in any other respect [Høyrup 2000a]. Together with the whole technique of *psēphos*-based reasoning it is thus almost certainly a borrowing from Near Eastern practical mathematicians.

[42] MS. VAT Lat. 4826, fol. 17r. I translate from my own transcription of the manuscript [1999].

> If some computation should be given to us in which three things were proposed, then we should always multiply the thing that we want to know against that which is not similar, and divide in the other thing, that is, in the other that remains.

After this follows a sequence of examples, beginning with this:

> I want to give you an example to the said rule, and I want to say thus, VII *tornesi* are worth VIIII *parigini*.[43] Say me, how much will 20 *tornesi* be worth? Do thus, the thing that you want to know is that which 20 *tornesi* will be worth. And the not similar (thing) is that which VII *tornesi* are worth, that is, they are worth 9 *parigini*. And therefore we should multiply 9 *parigini* times 20, they make 180 *parigini*, and divide in 7, which is the third thing. Divide 180, from which results 25 and $\frac{5}{7}$. And 25 *parigini* and $\frac{5}{7}$ will 20 *tornesi* be worth. And thus the similar computations are done.

This is the rule of three, and may be customary. But try to explain why it works without using paper and symbolic manipulations to somebody who is not too well trained in mathematics! The reason for the difficulty is of course that the intermediate result 9 *parigini* × 20 *tornesi* has no concrete interpretation.

Babylonian, Egyptian and ancient Greek calculators would have proceeded differently. Their normal procedure would have been to divide first (by whatever method they would use for division) 9 *parigini* by 7 *tornesi*. The result has an obvious concrete interpretation, the value of 1 *torneso* in *parigini*. Next, this could be multiplied by 20 in order to find the value of 20 *tornesi*.

Why was this easy and didactically efficient procedure given up? The key is inherent in the remark "by whatever method ...". Division is difficult, and often leads to rounding (either for reasons of principle, namely if you have to multiply by a reciprocal, or because it may lead to a very unhandy string of aliquot parts). Subsequent multiplication will also lead to multiplication of the rounding error, quite apart from the practical difficulty of multiplying an inconveniently composite numerical expression. Better therefore postpone the division and make it the last step.

Why, then, was it not given up before?[44] Once again, the explanation is straightforward and of a practical nature. It was set forth by Christian Wolff alias Doktor Pangloss in his *Mathematisches Lexikon* [1716: 867]:

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[43] The *parigino* and the *torneso* are coins, minted in Paris and Tours, respectively.

[44] In fact it was – but in India, where the characteristic terms of the rule of three can be traced back to c. 400 BCE [Sarma, forthcoming], and in China, where it is introduced in chapter 2 of the *Nine Chapters on Arithmetic* from the first century CE [trans. Vogel 1968: 18*ff*]. Medieval Islamic mathematicians (and probably practical reckoners) borrowed it from India.

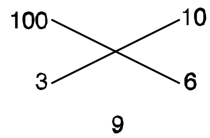
> It is true that performing mathematics can be learned without reasoning mathematics; but then one remains blind in all affairs, achieves nothing with suitable precision and in the best way, at times it may occur that one does not find one's way at all. Not to mention that it is easy to forget what one has learned, and that that which one has forgotten is not so easily retrieved, because everything depends only on memory

– in other words, only procedures that are performed so often that you run no risk of forgetting them (like changing gears in a car) can be safely taught as mere skills. Probably the scribes of Near Eastern Antiquity did not perform the kind of proportional operations we are speaking of so often that the appeal to their understanding could be given up safely.

More complex linear problems were often solved by means of the so-called "double false position", which is even more opaque. The intelligible alternative to this rule can be illustrated by another quotation from Jacopo (fol. 22r):

> I have new *fiorini* and old *fiorini*. And the *old fiorino* is worth *soldi* 35, and the new *fiorino* is worth *soldi* 37. And I have changed 100 *fiorini* now and old together, and I have got for them *libre* 178. I want to know how many new *fiorini* and how many old *fiorini* I had. Do thus, posit the case that all were of one of these rates, that is, all 100 of whatever rate you want. And let us say that they are all 100 old *fiorini*. And know how much they are worth for *soldi* 35 each, they are worth 175. Now say thus, from 175 until 178 there is *libre* 3, which are *soldi* 60. Now divide *soldi* 60 in the price difference which there is from one *fiorino* to the other, that is, from 35 *soldi* until 37, which is 2. Divide 60 in 2, 30 results. And 30 *fiorini* shall we say have been of the opposite (sort) of those {..} which we said were all old. And therefore we shall say that these 30 have been new, and the rest until 100, which is 70, have been old. And thus I say that they were.

This is easily understood (once you know that 1 *libra* is worth 20 *soldi*) – and precisely the same method (starting only from a fifty-fifty assumption) is used in the Old Babylonian problem VAT 8389 #1 [ed. Neugebauer 1935: I, 317*f*, III, 58]. If the double false position had been applied, the procedure had been much less comprehensible. One of false assumptions might be that all were old, in which case they would have been worth 3500 *soldi* = 175 *libre* – three less than I really get. The other false assumption might be that only 10 were old[45] and 90 hence new; in this case, I would have got 184 *libre*. The whole thing might be inserted in a graphical scheme



in which you were the to perform a cross-multiplication, add and divide by

[45] The Indians might have chosen that none were old, since they operated with both zero and negative numbers; but this simple choice was not accessible around the Mediterranean.

the sum of the two errors as written at bottom,[46] finding the real number of old *fiorini* to be $\frac{100\times6+10\times3}{9}$ = 70.

The principle can be explained as a linear interpolation; the real origin may be the alligation rule. But the texts never give any explanation, they simply set it forth as a rule to be followed. The obvious danger is that it may happen to be applied to non-linear situations, and that the reckoner would have no possibility to known that this was wrong.[47]

The moral is that Doktor Pangloss was right as soon as we get beyond the most routine applications of mathematics. A fundament in reason *is* an advantage not only in mathematical theory (where it belongs to the definition and is thus no mere advantage) but also in every application that goes beyond complete routine. It is therefore to be expected that the mathematics teaching in any mathematical culture which went beyond mere routine (on its own conditions for what could constitute routine) did include appeals to reason – whether naive or critical, and whether in Greek style (or that dubious reading of the Greek style in which we project ourselves) is a different matter. If we cannot find traces of this reasoning in extant sources we may safely conclude that this is due, *either* to failing understanding ofthe sources on our part, *or* to the insufficiency of extant sources as mirrors of educational practice. *Tertium non datur*.

46 Presupposing that one error is an excess, the other a deficit.

47 I am referring here to Mediterranean texts. Even though Arabic writers ascribe the rule to India, the simple form is not found in extant Indian sources. But what may be a correct iterated use in a non-linear situation turns up in a Sanskrit text from the fifteenth century [Plofker 1996; cf. *id.*, forthcoming] – if so, Indian reckoners knew what they were doing when applying the rule, and why.

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